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Shrinking Accelerated Forward-Backward-Forward Method for Split Equilibrium Problem and Monotone Inclusion Problem in Hilbert Spaces

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ABSTRACT

We propose and analyze a hybrid splitting method, comprises of forward-backward-forward iterates, shrinking projection iterates and Nesterov's acceleration method, to solve the monotone inclusion problem associated with maximal monotone operators and split equilibrium problem in Hilbert spaces. The proposed iterative method exhibits accelerated strong convergence characteristics under suitable set of control conditions in such framework. Finally, we explore some useful applications of the proposed iterative method via Numerical experiment.

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1. Introduction

The theory of convex optimization, as a subject, is developed to the discovery of problem arising in applied mathematics for which optimization algorithms or iterative methods are the effective tools. As a result, powerful optimization tools found valuable applications in core areas of applied mathematics as well as in automatic control systems, medicine, economics, signal processing, management, communications and networks, industry, combinatorial optimization, global optimization and other branches of sciences. Since convex optimization has a long historical roots, however, several recent developments in the subject not only stimulated the interest of researchers but also serve as an interdisciplinary bridge between various branches of sciences as mentioned above. It is therefore natural to recognize and formulate

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various real world and theoretical problems in the general framework of convex optimization to be solved numerically. Iterative methods are ubiquitous in the theory of convex optimization and still new iterative and theoretical techniques have been proposed and analyzed for convex optimization problems. Such an algorithm or iterative method is designed for the selection of the best out of many possible decisions in a real-life environment, constructing computational methods to find optimal solutions, exploring the theoretical properties and studying the computational performance of numerical algorithms implemented based on computational methods. Monotone operator theory is a fascinating field of research in nonlinear functional analysis. This class of operators attracts the research community primarily due to the importance of these operators in modelling problems in the field of convex optimization, subgradients, partial differential equations, variational inequalities and image processing, evolution equations and inclusions, see for instance, [12, 26, 30] and the references cited therein. Many problems in the theory of convex optimization concern with the approximation of zeroes of a maximal monotone operator defined on a Hilbert space. On the other hand, the problem of finding zeroes of the sum of two (maximal -) monotone operators is of fundamental importance in structured convex optimization, variational analysis, machine learning, signal processing and image analysis [22, 28]. Since the structured convex optimization problems are complex in nature and require sophisticated tools for the consequent analysis. Therefore, operator splitting technique is the most efficient tool to solve the structured convex optimization problem comprises of smooth and non-smooth functions. Moreover, operator splitting technique provides parallel computing architectures and thus reducing the complexity by splitting the original problem into simpler problems. The forward-backward (FB) algorithm is prominent among various splitting algorithms to find a zero of the sum of two maximal monotone operators [22]. Note that the FB algorithm efficiently tackle the situation for smooth and/or non-smooth functions. We remark that the several general splitting algorithms are available in the literature with specific limitations. However, new splitting algorithms are formulated in such a way to unify and/or combine the existing splitting algorithms with enhanced intrinsic properties. We, therefore, propose and analyze a splitting method comprises of forward-backward-forward (FBF) iterates in Hilbert spaces. In 1964, Polyak [29] employed the inertial extrapolation technique, based on the heavy ball methods of the two-order time dynamical system, to equip the iterative algorithm with fast convergence characteristic. It is remarked that the inertial term is computed by the difference of the two preceding iterations. The inertial extrapolation technique was originally proposed for minimizing differentiable convex functions, but it has been generalized in different ways. The heavy ball method has been incorporated in various iterative algorithms to obtain the fast convergence characteristic, see, for example [1] and the references cited therein. One of the main motivations for this paper is to equip the FBF algorithm with the inertial extrapolation technique for fast convergence results in Hilbert spaces. In order to ensure the strong convergence characteristics of the proposed algorithm, the shrinking effect of the half space is also employed in this framework. The theory of equilibrium problems is a systematic approach to study a diverse range of problems arising in the field of physics, optimization, variational inequalities, transportation, economics, network and noncooperative games, see, for example [13, 21] and the references cited therein. The existence result of an equilibrium problem can be found in the seminal work of Blum and Oettli [13]. Moreover, this theory has a computational flavor and flourishes significantly due to an excellent paper of Combettes and Hirstoaga [20]. The classical equilibrium problem theory has been generalized in several interesting ways to solve real world problems. In 2012, Censor et al. [18] proposed a theory regarding split variational inequality problem (SVIP) which aims to solve a pair of

variational inequality problem in such a way that the solution of a variational inequality problem, under a given bounded linear operator, solves another variational inequality.
Motivated by the work of Censor et al. [18], Moudafi [27] generalized the concept of SVIP to that of split monotone variational inclusions (SMVIP) which includes, as a special case, split variational inequality problem, split common fixed point problem, split zeroes problem, split equilibrium problem and split feasibility problem. These problems have already been studied and successfully employed as a model in intensity-modulated radiation therapy treatment planning, see [16,17]. This formalism is also at the core of modeling of many inverse problems arising for phase retrieval and other real-world problems; for instance, in sensor networks in computerized tomography and data compression; see, for example, [19]. Some methods have been proposed and analyzed to solve split equilibrium problem and mixed split equilibrium problem in Hilbert spaces, see, for example, [2–11] and the references cited therein. Inspired and motivated by the above mentioned results and the ongoing research in this direction, we aim to employ a hybrid splitting method, comprises of forward-backward-forward iterates, shrinking projection iterates and Nesterov's acceleration method, to solve the monotone inclusion problem associated with maximal monotone operators and split equilibrium problem in Hilbert spaces. The proposed iterative method exhibits accelerated strong convergence characteristics under suitable set of control conditions in such framework. Finally, we explore some useful applications of the proposed iterative method via numerical simulation. The rest of the paper is organized as follows: Section 2 contains preliminary concepts and results regarding monotone operator theory and equilibrium problem theory. Section 3 comprises of strong convergence results of the proposed algorithm. Section 4 deals with applications of (FBF) method in minimization problem, split feasibility problem, monotone variational inequality problem and Image processing. Section 5 deals with the efficiency of the proposed algorithm and its comparison with the existing algorithm by numerical experiments.

2. Preliminaries

Throughout this section, we first fix some necessary notions and concepts which will be required in the sequel(see [12] for a detailed account). We denote by $\mathbb{N}$ the set of all natural numbers and $\mathbb{R}$ the set of all real numbers, respectively. Let $C \subseteq \mathcal{H}_1$ and $Q \subseteq \mathcal{H}_2$ be two non-empty subsets of real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ with the inner product $\langle \cdot\,, \cdot \rangle$ and the associated norm $\|\cdot\|$. Let $x_n \to x$ (resp. $x_n \rightharpoonup x$) indicates strong convergence (resp. weak convergence) of a sequence $\{x_n\}_{n=1}^{\infty}$ in C. Let $A : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ be an operator. We denote $dom\,(A) = \{x \in \mathcal{H}_1 : Ax \neq \emptyset\}$ the domain of A, $Gr(A) = \{(x, u) \in \mathcal{H}_1 \times \mathcal{H}_1 : u \in Ax\}$ the graph of A and $zer\,(A) = \{x \in \mathcal{H}_1 : 0 \in Ax\}$ the set of zeros of A. The inverse of A, that is, A^{-1} is defined as $(u, x) \in Gr(A^{-1})$ if and only if $(x, u) \in Gr(A)$ and the resolvent of A is denoted as $J_A = (I + A)^{-1}$. It is remarked that $J_A : \mathcal{H}_1 \to \mathcal{H}_1$ is single valued and maximal monotone operator provided that A is maximal monotone. Recall that A is said to be: (i) monotone if $\langle x - y, u - v\rangle \geq 0$ for all $(x, u), (y, v) \in Gr(A)$; (ii) maximally monotone if A is monotone and there exists no monotone operatort $B : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ such that $Gr(B)$ properly contains the $Gr(A)$; (iii) strongly monotone with modulus $\alpha > 0$ such that $\langle x - y, u - v\rangle \geq \alpha \|x - y\|^2$ for all $(x, u), (y, v) \in Gr(A)$ and (iv) inverse strongly monotone (cocoercive) with parameter β such that $\langle x - y, Ax - Ay\rangle \geq \beta \|Ax - Ay\|^2$.

Let $f : \mathcal{H}_1 \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function and let $g : \mathcal{H}_1 \to \mathbb{R}$ be a convex, differentiable and Lipschitz continuous gradient function, then the convex minimization problem for f and g is defined as:

$$\min_{x\in\mathcal{H}_1}\{f(x)+g(x)\}. \tag{2.1}$$

The subdifferential of a function f is defined and denoted as:

$$\partial f(x)=\{x^*\in\mathcal{H}_1 : f(y)\geq f(x)+\langle x^*,y-x\rangle \text{ for all } x\in\mathcal{H}_1\}.$$

It is remarked that the subdifferential of a proper convex lower semicontinuous function is a maximally monotone operator. The proximity operator of a function f is defined as:

$$\text{prox}_f:\mathcal{H}_1\to\mathcal{H}_1 : x\mapsto \underset{y\in\mathcal{H}_1}{\text{argmin}}\left(f(y)+\frac{1}{2}\|x-y\|^2\right).$$

Note that the proximity operator is linked with the subdifferential operator such that argmin $(f)=zer(\partial f)$. Moreover, $\text{prox}_f=J_{\partial f}$. Utilizing the said connection, we state monotone inclusion problem with respect to a maximally monotone operator A and an arbitrary operator B is to find:

$$x^*\in C \quad \text{such that } 0\in Ax^*+Bx^*. \tag{2.2}$$

The solution set of the problem (2.1) is denoted as $zer(A+B)$.

We now define the concept of (mixed) split equilibrium problem.

Let $h:\mathcal{H}_1\to\mathcal{H}_2$ be a bounded linear operator. Let $F:C\times C\to\mathbb{R}$ and $G:Q\times Q\to\mathbb{R}$ be two bifunctions, $\phi_f:C\to\mathcal{H}_1$ and $\phi_g:Q\to\mathcal{H}_2$ be two nonlinear operators. Recall that a split equilibrium problem (SEP) is to find:

$$x^*\in C \quad \text{such that } F(x^*,x)\geq 0 \text{ for all } x\in C, \tag{2.3}$$

and

$$y^*=hx^*\in Q \quad \text{such that } G(y^*,y)\geq 0 \text{ for all } y\in Q. \tag{2.4}$$

The solution set of the split equilibrium problem (2.2) and (2.3) is denoted by

$$SEP(F):=\{x^*\in C : x^*\in EP(F) \quad \text{and} \quad hx^*\in EP(G)\}. \tag{2.5}$$

Let C be a nonempty closed convex subset of a Hilbert space $\mathcal{H}_1$. Let C be a nonempty closed convex subset of a Hilbert space $\mathcal{H}_1$. For each $x\in\mathcal{H}_1$, there exists a unique nearest point of C, denoted by P_Cx, such that

$$\|x-P_Cx\|\leq\|x-y\| \text{ for all } y\in C.$$

Such a mapping $P_C:\mathcal{H}_1\to C$ is known as a metric projection or a nearest point projection of $\mathcal{H}_1$ onto C. Moreover, P_C satisfies nonexpansiveness in a Hilbert space and $\langle x-P_Cx,P_Cx-y\rangle\geq 0$ for all $x,y\in C$. It is remarked that P_C is firmly nonexpansive mapping from $\mathcal{H}_1$ onto C, that is,

$$\|P_Cx-P_Cy\|^2\leq\langle x-y,P_Cx-P_Cy\rangle, \text{ for all } x,y\in C.$$

Moreover, for any $x\in\mathcal{H}_1$ and $z\in C$, we have $z=P_Cx$ if and only if $\langle x-z,z-y\rangle\geq 0$ for every $y\in C$. The following lemma collects some well-known results in the context of a real Hilbert space.

The following lemma collects some well-known results in the context of a real Hilbert space.

Lemma 2.1. [12] *The following properties hold in a real Hilbert space* $\mathcal{H}_1$*:*

1. $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$, *for all* $x, y \in \mathcal{H}_1$;
2. $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$, *for all* $x, y \in \mathcal{H}_1$;
3. $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$, *for every* $x, y \in \mathcal{H}_1$ *and* $\mu \in [0,1]$.

Lemma 2.2. [14] *Let* T *be an nonexpansive mapping with* $F(T) \neq \emptyset$ *defined it as for a closed nonempty convex subset* $\widehat{C}$ *of a real Hilbert space* $\mathcal{H}_1$*. we assume that* $\{u_n\}$ *is a sequence in* $\widehat{C}$ *such that* $u_n \rightharpoonup u$ *and* $(I-T)u_n \to v$*, then* $(I-T)u = v$*. In particular, if* $v = 0$*, then* $u \in F(T)$.

Asumption 2.3. *Let* C *be a nonempty closed and convex subset of a Hilbert space* $\mathcal{H}_1$*. Let* $F : C \times C \to \mathbb{R}$ *be a bifunction and lower semicontinuous satisfying the following conditions:*

(A1) $F(x,x) = 0$ *for all* $x \in C$;
(A2) F *is monotone, i.e.,* $F(x,y) + F(y,x) \le 0$ *for all* $x, y \in C$;
(A3) *for each* $x, y, z \in C$, $\limsup_{t\to 0} F(tz + (1-t)x, y) \le F(x,y)$;
(A4) *for each* $x \in C$, $y \mapsto F(x,y)$ *is convex and lower semi-continuous.*

Lemma 2.4. [24] *Let* C *be a closed convex subset of a real Hilbert space* $\mathcal{H}_1$ *and let* $F : C \times C \to \mathbb{R}$ *be a bifunction satisfying conditions (A1)-(A4) of Assumption 2.3. For* $r > 0$ *and* $x \in \mathcal{H}_1$*, there exists* $z \in C$ *such that*

$$F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \ge 0, \text{ for all } y \in C.$$

Moreover, define a mapping $T_r^F : \mathcal{H}_1 \to C$ *by*

$$T_r^F(x) = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \ge 0, \quad \text{for all } y \in C \right\},$$

for all $x \in \mathcal{H}_1$*. Then, the following results hold:*

(1) *for each* $x \in \mathcal{H}_1$, $T_r^F \neq \emptyset$;
(2) T_r^F *is single-valued;*
(3) T_r^F *is firmly nonexpansive, i.e.,*
$$\text{for every } x, y \in \mathcal{H}_1, \left\| T_r^F x - T_r^F y \right\|^2 \le \left\langle T_r^F x - T_r^F y, x-y \right\rangle;$$
(4) $F(T_r^F) = SEP(F)$;
(5) $SEP(F)$ *is closed and convex.*

It is remarked that if $G : Q \times Q \to \mathbb{R}$ *is a bifunction satisfying conditions (A1)-(A4), wher* Q *is a nonempty closed and convex subset of a Hilbert space* $\mathcal{H}_2$*. Then for each* $s > 0$ *and* $w \in \mathcal{H}_2$ *we can define a mapping:*

$$T_s^G(w) = \left\{ d \in C : G(d,e) + \frac{1}{s}\langle e-d, d-w\rangle \ge 0, \quad \text{for all } e \in Q \right\},$$

which is, nonempty, single-valued and firmly nonexpansive. Then the following results hold:

(1) *for each* $w \in \mathcal{H}_2$, $T_s^G \neq \emptyset$;
(2) T_s^G *is single-valued;*
(3) T_s^G *is firmly nonexpansive;*
(4) $F(T_s^G) = SEP(G)$;
(5) $SEP(G)$ *is closed and convex.*

Lemma 2.5. [31] *Let E be a Banach space satisfying Opial's condition and let $\{x_n\}$ be a sequence in E. Let $l, m \in E$ be such that $\lim_{n\to\infty} \|x_n - l\|$ and $\lim_{n\to\infty} \|x_n - m\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to l and m, respectively, then $l = m$.*

Lemma 2.6. [23] *Let $A : E \to E$ be an γ-inverse strongly accretive of order r and $B : E \to 2^E$ an n-accretive operator,where E be a Banach space. Then the following inequalities holds:*
a) *For $c > 0$, $F(T_c^{A,B}) = (A + B)^{-1}(0)$.*
b) *For $0 < d \leq c$ and $u \in E, \|u - T_d^{A,B} u\| \leq 2\|u - T_c^{A,B}\|$.*

Lemma 2.7. [25] *Let $\widehat{C}$ be a closed nonempty and convex subset of a real Hilbert space $\mathcal{H}_1$. For every $r, s \in \mathcal{H}_1$ and $\gamma \in \mathbb{R}$ the set*
$D = \{u \in C : \|s - u\|^2 \leq \|r - u\|^2 + \langle z, u\rangle + \gamma\}$ is convex and closed.

Lemma 2.8. *Let $P_{\widehat{C}} : \mathcal{H} \to \widehat{C}$ be the metric projection from $\mathcal{H}$ onto $\widehat{C}$. Then the following inequality satisfied:*

$$\|v - P_{\widehat{C}}u\|^2 + \|u - P_{\widehat{C}}u\|^2 \leq \|u - v\|^2,$$

for all $u \in \mathcal{H}$ and for all $v \in \widehat{C}$ and $\widehat{C}$ be a closed nonempty and convex subset of a real Hilbert space $\mathcal{H}$.

3. Main results

In this section, we prove some strong convergence theorems of an inertial method with a forward-backward-forward splitting algorithm for solving the split equilibrium problem together with the monotone inclusion problem in the framework of Hilbert spaces. Now, We prove the following strong convergence Theorem.

Theorem 3.1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be two bifunctions satisfying Assumption 2.3 such that G is upper semicontinuous. Let $h : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator; let $A : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ be a maximally monotone operator and let $B : \mathcal{H}_1 \to \mathcal{H}_1$ be a monotone and ρ-Lipschitz operator for some $\rho > 0$. Assume that $\Gamma = (A + B)^{-1}(0) \cap \Omega \neq \emptyset$, where $\Omega := \{x^* \in C : x^* \in EP(F)$ and $hx^* \in EP(G)\}$, $\{\mu_n\} \subset (0, \frac{2}{\rho})$, $\{\gamma_n\} \subset [0, \gamma]$, $\gamma \in [0, \frac{1}{2})$, $\{r_n\} \subset (0, \infty)$, with $\alpha \in (0, \frac{1}{L})$ such that L is the spectral radius of h^*h and $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0, 1]$. For given $x_0, x_1 \in \mathcal{H}_1$ let the iterative sequence $\{x_n\}$,$\{m_n\}$,$\{u_n\}$,$\{v_n\}$ and $\{w_n\}$ be generated by*

$$\begin{cases} m_n = x_n + \gamma_n(x_n - x_{n-1}); \\ u_n = \beta_n m_n + (1 - \beta_n)T_{r_n}^F(I - \alpha h^*(I - T_{r_n}^G)h)m_n; \\ v_n = J_{\mu_n A}(I - \mu_n B)u_n; \\ w_n = v_n - \mu_n(Bv_n - Bu_n); \\ C_{n+1} = \{\widehat{u} \in C_n : \|w_n - \widehat{u}\|^2 \leq \|x_n - \widehat{u}\|^2 - 2\gamma_n\langle x_n - \widehat{u}, x_{n-1} - x_n\rangle + \gamma_n^2\|x_{n-1} - x_n\|^2\}; \\ x_{n+1} = P_{C_n}(x_0). \end{cases} \tag{3.1}$$

Assume that the following conditions hold:

C1 $\sum_{n=1}^{\infty} \gamma_n\|x_n - x_{n-1}\| < \infty$;

C2 $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

C3 $\liminf_{n\to\infty} r_n > 0$;

C4 $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < \frac{2}{\rho}$.

Then the sequence $\{x_n\}$ generated by (3.1) strongly convergence to a point $\widehat{u} = P_\Gamma x_1$.

Proof. We solve this theorem by dividing it into six steps.
Step 1. Show that $P_{C_{n+1}}x_1$ is well-defined for every $x \in \mathcal{H}_1$. We know that $(A+B)^{-1}(0)$ and Ω are closed and convex by Lemma 2.4 and 2.6, respectively. From the definition of C_{n+1} and from Lemma 2.9 C_{n+1} is closed and convex for each $n \geq 1$. For each $n \in \mathbb{N}$ and let $\widehat{u} \in \Gamma$. Since we write $T_n = T^F_{r_n}(I - \alpha h^*(I - T^G_{r_n})h)$. Note that T_n is a quasi-nonexpansive mapping and $J_{\mu_n A}$ is nonexpansive. For simplicity of the algorithm, we write $w_n = \beta_n v_n + (1-\beta_n)T_n v_n$ and for every $n \in \mathbb{N}$. We have

$$\begin{aligned}\|m_n - \widehat{u}\|^2 &= \|x_n - \widehat{u} - \gamma_n(x_{n-1} - x_n)\|^2 \\ &= \|x_n - \widehat{u}\|^2 + \gamma_n^2\|x_{n-1} - x_n\|^2 - 2\gamma_n\langle x_n - \widehat{u}, x_{n-1} - x_n\rangle. \end{aligned} \tag{3.2}$$

Further,

$$\begin{aligned}\|u_n - \widehat{u}\|^2 &= \|\alpha_n m_n + (1-\alpha_n)T_n m_n - \widehat{u}\|^2 \\ &= \|\alpha_n(m_n - \widehat{u}) + (1-\alpha_n)(m_n - \widehat{u})\|^2 \\ &= \|m_n - \widehat{u}\|^2 \\ &= \|x_n - \widehat{u}\|^2 + \gamma_n^2\|x_{n-1} - x_n\|^2 - 2\gamma_n\langle x_n - \widehat{u}, x_{n-1} - x_n\rangle. \end{aligned} \tag{3.3}$$

Furthermore,

$$\begin{aligned}\|v_n - \widehat{u}\|^2 &= \|J_{\mu_n A}(I - \mu_n B)u_n - J_{\mu_n A}(I - \mu_n B)\widehat{u}\|^2 \\ &= \|u_n - \mu_n Bu_n - (\widehat{u} - \mu_n B\widehat{u})\|^2 \\ &= \|u_n - \widehat{u} - (\mu_n Bu_n - \mu_n B\widehat{u})\|^2 \\ &\leq \|u_n - \widehat{u}\|^2 + \mu_n^2\|Bu_n - B\widehat{u})\|^2 - 2\mu_n\langle u_n - \widehat{u}, Bu_n - B\widehat{u}\rangle \\ &\leq \|u_n - \widehat{u}\|^2 + \mu_n^2\|Bu_n - B\widehat{u})\|^2 - \frac{2\mu_n}{\rho}\|Bu_n - B\widehat{u}\|^2 \\ &\leq \|u_n - \widehat{u}\|^2 + \mu_n(\mu_n - \frac{2}{\rho})\|Bu_n - B\widehat{u})\|^2 \\ &= \|u_n - \widehat{u}\|^2 \\ &\leq \|x_n - \widehat{u}\|^2 + \gamma_n^2\|x_{n-1} - x_n\|^2 - 2\gamma_n\langle x_n - \widehat{u}, x_{n-1} - x_n\rangle. \end{aligned} \tag{3.4}$$

Note that, if $\widehat{u} \in zer(A+B)$, then $(Bx_n)_{n\in\mathbb{N}}$ converges strongly to the unique dual solution

Bx [see proof: Theorem 26.14(ii) of [12]], so therefore, $Bu_n - B\widehat{u} \to 0$. So we observe that,

$$\begin{aligned}
\|w_n - \widehat{u}\|^2 &= \|v_n - \mu_n(Bv_n - Bu_n) - (\widehat{u} + B\widehat{u} - B\widehat{u})\|^2 \\
&= \|(v_n - \widehat{u}) - \mu_n(Bv_n - Bu_n) - (B\widehat{u} - B\widehat{u})\|^2 \\
&= \|(v_n - \widehat{u}) - \mu_n(Bv_n - B\widehat{u}) + \mu_n(Bu_n - B\widehat{u})\|^2 \\
&= \|(v_n - \widehat{u}) - \mu_n(Bv_n - B\widehat{u})\|^2 + \mu_n^2\|(Bu_n - B\widehat{u})\|^2 \\
&\quad +2\mu_n\langle(v_n - \widehat{u}) - \mu_n(Bv_n - B\widehat{u}), Bu_n - B\widehat{u}\rangle \\
&= \|(v_n - \widehat{u}) - \mu_n^2(Bv_n - B\widehat{u})\|^2 + \mu_n^2\|Bu_n - B\widehat{u}\|^2 \\
&\quad +2\mu_n\langle v_n - \widehat{u}, Bu_n - B\widehat{u}\rangle - 2\mu_n^2\langle Bv_n - B\widehat{u}, Bu_n - B\widehat{u}\rangle \\
&= \|(v_n - \widehat{u}) - \mu_n^2(Bv_n - B\widehat{u})\|^2 \\
&= \|v_n - \widehat{u}\|^2 + \mu_n^2\|Bv_n - B\widehat{u}\|^2 - 2\mu_n\langle v_n - \widehat{u}, Bv_n - B\widehat{u}\rangle \\
&= \|v_n - \widehat{u}\|^2 + \mu_n(\mu_n - \frac{2}{\rho})\|Bv_n - Bx\|^2 \\
&= \|v_n - \widehat{u}\|^2 \\
&= \|x_n - \widehat{u}\|^2 + \gamma_n^2\|x_{n-1} - x_n\|^2 - 2\gamma_n\langle x_n - \widehat{u}, x_{n-1} - x_n\rangle. \qquad (3.5)
\end{aligned}$$

$\widehat{u} \in C_n$, for all $n \geq 1$. Hence $\widehat{u} \in C_{n+1}$ implies $\Gamma \subset C_{n+1}$. Therefore $P_{C_{n+1}}x_1$ is well defined.
Step 2. Next we show that $\lim_{n\to\infty} \|x_n - \widehat{u}\|$ exists. Since Γ is nonempty, closed and convex subset of $\mathcal{H}_1$, there exist a unique $x^* \in \Gamma$ such that

$$x^* = P_\Gamma x_1.$$

From $P_{C_{n+1}}x_1$, $C_{n+1} \subset C_n$ and $x_{n+1} \in C_{n+1}$, for all $n \geq 1$, we get

$$\|x_n - \widehat{u}\| \leq \|x_{n+1} - \widehat{u}\|, \text{for all } n \geq 1.$$

In either case, as $\Gamma \subset C_n$, we have

$$\|x_n - \widehat{u}\| \leq \|x^* - \widehat{u}\|, \text{for all } n \geq 1. \qquad (3.6)$$

This implies that $\{x_n\}$ is bounded, nondecreasing and well defined, hence

$$\lim_{n\to\infty} \|x_n - \widehat{u}\| \qquad \text{exists.} \qquad (3.7)$$

Step 3. Next, we show that $x_n \to \widehat{u} \in C$ as $n \to \infty$. For $m > n$, by the definition of C_n, we have $x_m = P_{C_m}x_1 \in C_m \subseteq C_n$. By Lemma 2.8 we estimate that,

$$\|x_m - x_n\|^2 \leq \|x_m - \widehat{u}\|^2 - \|x_n - \widehat{u}\|^2. \qquad (3.8)$$

Since, $\lim_{n\to\infty} \|x_n - \widehat{u}\|$ exists, it follows from (3.8) that $\lim_{n\to\infty} \|x_m - x_n\| = 0$. Hence, $\{x_n\}$ is a Cauchy sequence in C and $x_n \to \widehat{u} \in C$ as $n \to \infty$.
Step 4. Show that $\lim_{n\to\infty} x_n = \widehat{u}$, where $\widehat{u} = P_\Gamma(x_0)$. we obtain from (3.1) and Step 3 that, let $\widehat{u} \in zer(A + B)$, that is $-B\widehat{u} \in A\widehat{u}$. According to the definition of the resolvent, we have

$$\|m_n - x_n\| = |\gamma_n|\|x_n - x_{n-1}\| = 0. \qquad (3.9)$$

as $n \to +\infty$, $\|x_n - x_{n-1}\| \to 0$.

From (3.1), we get,

$$\begin{aligned} \|u_n - x_n\|^2 &= \|\alpha_n m_n + (1-\alpha_n) T_n m_n - x_n\|^2 \\ &\leq \|\alpha_n (m_n - x_n) + (1-\alpha_n)(m_n - x_n)\|^2 \\ &\leq \|m_n - x_n\|^2. \end{aligned} \tag{3.10}$$

Hence,

$$\|x_{n+1} - m_n\| \leq \|x_{n+1} - x_n\| + \|m_n - x_n\| \to 0 \quad as \quad n \to \infty.$$

Since, $x_{n+1} \in C_n$, we obtain,

$$\begin{aligned} \|v_n - x_{n+1}\|^2 &\leq \|x_n - x_{n+1}\|^2 - 2\gamma_n \langle x_n - x_{n+1}, x_{n-1} - x_n \rangle + \gamma_n^2 \|x_{n-1} - x_n\|^2 \\ &\leq \|x_n - x_{n+1}\|^2 + 2|\gamma_n| \|x_n - x_{n+1}\| \|x_{n-1} - x_n\| \\ &\quad + \gamma_n^2 \|x_{n-1} - x_n\|^2 \to 0, \quad as \quad n \to \infty. \end{aligned} \tag{3.11}$$

for all $n \geq 1$. So,

$$\lim_{n\to\infty} \|v_n - x_n\| = 0. \tag{3.12}$$

and

$$\begin{aligned} \|w_n - x_n\|^2 &= \|v_n - x_n - \mu_n (Bv_n - Bu_n)\|^2 \\ &\leq \|v_n - x_n\|^2 + \mu_n^2 \|Bv_n - Bu_n\|^2 - \frac{2\mu_n}{\rho} \|Bv_n - Bu_n\|^2 \\ &\leq \|v_n - x_n\|^2 + \mu_n (\mu_n - \frac{2}{\rho}) \|Bv_n - Bu_n\|^2 \\ &\leq \|v_n - x_n\|^2 \to 0, asn \to \infty. \end{aligned} \tag{3.13}$$

Similarly,

$$\|w_n - v_n\| \leq \|w_n - x_n\| + \|v_n - x_n\| \to 0 \quad as \quad n \to \infty.$$

$$\|v_n - u_n\| \leq \|v_n - x_n\| + \|u_n - x_n\| \to 0 \quad as \quad n \to \infty.$$

$$\|u_n - m_n\| \leq \|u_n - x_n\| + \|m_n - x_n\| \to 0 \quad as \quad n \to \infty.$$

and

$$\|w_n - m_n\| \leq \|w_n - m_n\| + \|m_n - x_n\| \to 0 \quad as \quad n \to \infty. \tag{3.14}$$

Take $w_n := S_n m_n$, where $S_n := (I + \mu_n A)^{-1}(I - \mu_n B)$. Therefore,

$$\|S_n m_n - m_n\| = \|w_n - m_n\| \to 0 \quad as \quad n \to \infty.$$

Since $\liminf_{n\to\infty} \mu_n > 0$, there exist $\sigma > 0$ such that $\mu_n \geq \sigma$, for all $n \geq 1$. Then, by Lemma 2.6, we have,

$$\lim_{n\to\infty} \|S_\sigma m_n - m_n\| \leq 2 \lim_{n\to\infty} \|S_n m_n - m_n\| = 0.$$

By lemma 3.3 and 3.1 of [23], S_σ is nonexpansive and $F(S_\sigma) = (A+B)^{-1}(0)$. Since $\{x_n\}$ is bounded and $\mathcal{H}_1$ is reflexive, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \widehat{w} \in \mathcal{H}_1$. Using the fact that $\|u_n - x_n\| \to 0, \quad as \quad n \to \infty$ and $x_{n_i} \rightharpoonup \widehat{w} \in \mathcal{H}_1$, we have $u_{n_i} \rightharpoonup \widehat{w} \in \mathcal{H}_1$.

We can therefore make use of Lemma 2.2 to assure that $\widehat{w} \in \Gamma$. If $\widehat{u} = P_\Gamma(x_0)$, it follows from (3.7), the fact that $\widehat{w} \in \Gamma$ and the lower semicontinuity of the norm that,

$$\begin{aligned} \|x_0 - \widehat{u}\| &\leq \|x_0 - \widehat{w}\| \leq \liminf_{i\to\infty} \|x_0 - x_{n_i}\| \\ &\leq \limsup_{i\to\infty} \|x_0 - x_{n_i}\| \leq \|x_0 - \widehat{u}\|. \end{aligned}$$

Thus, we have that $\lim_{i\to\infty} \|x_{n_i} - x_0\| = \|x_0 - \widehat{w}\| = \|x_0 - \widehat{u}\|$.This implies that $x_{n_i} \to \widehat{w} = \widehat{u}, i \to \infty$. It follows that $\{x_n\}$ converges weakly to $\widehat{u}$. So we have,

$$\begin{aligned} \|x_0 - \widehat{u}\| &\leq \liminf_{n\to\infty} \|x_0 - x_n\| \\ &\leq \limsup_{n\to\infty} \|x_0 - x_n\| \leq \|x_0 - \widehat{u}\|. \end{aligned}$$

This shows that, $\lim_{n\to\infty} \|x_n - x_0\| = \|x_0 - \widehat{u}\|$. From $x_n \rightharpoonup \widehat{u}$, we also have $x_n - x_0 \rightharpoonup \widehat{u} - x_0$. Since $\mathcal{H}_1$ satisfies the Kadec-Klee property, it follows that $x_n - x_0 \to \widehat{u} - x_0$.
Therefore, $x_n \to \widehat{u}$ *as* $n \to \infty$.
Step 5. First we show that $h^*(I - T^G_{r_n})h$ is a $\frac{1}{L}$-inverse strongly monotone mapping. For this, we utilize the firmly nonexpansive of $T^G_{r_n}$ which implies that $(I - T^G_{r_n})$ is a 1-inverse strongly monotone mapping. Now, observe that

$$\begin{aligned} \|h^*(I - T^G_{r_n})hx - h^*(I - T^G_{r_n})hy\|^2 &= \langle h^*(I - T^G_{r_n})(hx - hy), h^*(I - T^G_{r_n})(hx - hy)\rangle \\ &= \langle (I - T^G_{r_n})(hx - hy), h^*h(I - T^G_{r_n})(hx - hy)\rangle \\ &\leq L\langle (I - T^G_{r_n})(hx - hy), (I - T^G_{r_n})(hx - hy)\rangle \\ &= L\|(I - T^G_{r_n})(hx - hy)\|^2 \\ &\leq L\langle x - y, h^*(I - T^G_{r_n})(hx - hy)\rangle, \end{aligned}$$

for all $x, y \in \mathcal{H}_1$. So, we observe that, $h^*(I - T^G_{r_n})h$ is a $\frac{1}{L}$-inverse strongly monotone. Moreover,$I - \alpha h^*(I - T^G_{r_n})h$ is nonexpansive provided $\alpha \in (0, \frac{1}{L})$.

Next, we show that $\widehat{v} \in \Omega$. Setting, $z_n = T^F_{r_n}(I - \alpha h^*(I - T^G_{r_n})h)m_n$. For any $\widehat{u} \in \Gamma$, we consider the following estimate:

$$\begin{aligned} \|z_n - \widehat{u}\|^2 &= \|T^F_{r_n}(I - \alpha h^*(I - T^G_{r_n})h)m_n - \widehat{u}\|^2 \\ &= \|T^F_{r_n}(I - \alpha h^*(I - T^G_{r_n})h)m_n - T^F_{r_n}\widehat{u}\|^2 \\ &\leq \|m_n - \alpha h^*(I - T^G_{r_n})hm_n - \widehat{u}\|^2 \\ &\leq \|m_n - \widehat{u}\|^2 + \alpha^2\|h^*(I - T^G_{r_n})hm_n\|^2 \\ &\quad +2\alpha\langle \widehat{u} - m_n, h^*(I - T^G_{r_n})hm_n\rangle. \end{aligned} \tag{3.15}$$

Thus, we have

$$\begin{aligned} \|z_n - \widehat{u}\|^2 &\leq \|x_n - \widehat{u}\|^2 + \gamma_n^2\|x_{n-1} - x_n\|^2 - 2\gamma_n\langle x_n - u, x_{n-1} - x_n\rangle \\ &\quad +\alpha^2\langle hm_n - T^G_{r_n}hm_n, h^*h(I - T^G_{r_n})hm_n\rangle \\ &\quad +2\alpha\langle \widehat{u} - m_n, h^*(I - T^G_{r_n})hm_n\rangle. \end{aligned} \tag{3.16}$$

Moreover, we have

$$\begin{aligned}\alpha^2\langle hm_n - T^G_{r_n}hm_n, h^*h(I - T^G_{r_n})hm_n\rangle &\leq L\alpha^2\langle hm_n - T^G_{r_n}hm_n, hm_n - T^G_{r_n}hm_n\rangle \\ &= L\alpha^2\|hm_n - T^G_{r_n}hm_n\|^2. \qquad (3.17)\end{aligned}$$

Note that

$$\begin{aligned}2\alpha\langle\widehat{u} - m_n, h^*(I - T^G_{r_n})hm_n\rangle &= 2\alpha\langle h(\widehat{u} - m_n), hm_n - T^G_{r_n}hm_n\rangle \\ &= 2\alpha\langle h(\widehat{u} - m_n), (hm_n - T^G_{r_n}hm_n) \\ &\quad -(hm_n - T^G_{r_n}hm_n), hm_n - T^G_{r_n}hm_n\rangle \\ &= 2\alpha[\langle Ap - T^G_{r_n}hm_n, hm_n - T^G_{r_n}hm_n\rangle - \|hm_n - T^G_{r_n}hm_n\|^2] \\ &\leq 2\alpha[\frac{1}{2}\|hm_n - T^G_{r_n}hm_n\|^2 - \|hm_n - T^G_{r_n}hm_n\|^2] \\ &= -\alpha\|hm_n - T^G_{r_n}hm_n\|^2. \qquad (3.18)\end{aligned}$$

Utilizing (3.16)-(3.18), we have

$$\begin{aligned}\|z_n - \widehat{u}\|^2 &\leq \|x_n - \widehat{u}\|^2 + \gamma_n^2\|x_{n-1} - x_n\|^2 - 2\gamma_n\langle x_n - u, x_{n-1} - x_n\rangle \\ &\quad + L\alpha^2\|hm_n - T^G_{r_n}hm_n\|^2 - \alpha\|hm_n - T^G_{r_n}hm_n\|^2 \\ &= \|x_n - \widehat{u}\|^2 + \gamma_n^2\|x_{n-1} - x_n\|^2 - 2\gamma_n\langle x_n - u, x_{n-1} - x_n\rangle \\ &\quad + \alpha(L\alpha - 1)\|hm_n - T^G_{r_n}hm_n\|^2. \qquad (3.19)\end{aligned}$$

Note that

$$\begin{aligned}\|u_n - \widehat{u}\|^2 &\leq \beta_n\|m_n - \widehat{u}\|^2 + (1 - \beta_n)\|z_n - \widehat{u}\|^2 \\ &\leq \|x_n - \widehat{u}\|^2 + \gamma_n^2\|x_{n-1} - x_n\|^2 - 2\gamma_n\langle x_n - u, x_{n-1} - x_n\rangle \\ &\quad + \alpha(L\alpha - 1)\|hm_n - T^G_{r_n}hm_n\|^2. \qquad (3.20)\end{aligned}$$

Moreover, we have

$$\begin{aligned}-\alpha(L\alpha - 1)\|hm_n - T^G_{r_n}hm_n\|^2 &\leq \|x_n - \widehat{u}\|^2 - \|u_n - \widehat{u}\|^2 + \gamma_n^2\|x_{n-1} - x_n\|^2 \\ &\quad - 2\gamma_n\langle x_n - u, x_{n-1} - x_n\rangle. \qquad (3.21)\end{aligned}$$

Since $\alpha(L\alpha - 1) < 0$, it follows from (3.7), C1 and the above estimate that

$$\lim_{n\to\infty}\|hm_n - T^G_{r_n}hm_n\| = 0. \qquad (3.22)$$

Note that $T_{r_n}^F$ is firmly nonexpansive and $I-\alpha h^*(I-T_{r_n}^G)h$ is nonexpansive, it follows that

$$\begin{aligned}
\|z_n-\widehat{u}\|^2 &= \|T_{r_n}^F(m_n-\alpha h^*(I-T_{r_n}^G)hm_n)-T_{r_n}^F\widehat{u}\|^2\\
&\leq \langle T_{r_n}^F(m_n-\alpha h^*(I-T_{r_n}^G)hm_n)-T_{r_n}^F\widehat{u}, m_n-\alpha h^*(I-T_{r_n}^G)hm_n-\widehat{u}\rangle\\
&= \langle z_n-\widehat{u}\ ,\ m_n-\alpha h^*(I-T_{r_n}^G)hm_n-\widehat{u}\rangle\\
&= \frac{1}{2}\{\|z_n-\widehat{u}\|^2+\|m_n-\alpha h^*(I-T_{r_n}^G)hm_n-\widehat{u}\|^2\\
&\quad -\|z_n-m_n+\alpha h^*(I-T_{r_n}^G)hm_n\|^2\}\\
&\leq \frac{1}{2}\{\|z_n-\widehat{u}\|^2+\|m_n-\widehat{u}\|^2-\|z_n-m_n+\alpha h^*(I-T_{r_n}^G)hm_n\|^2\}\\
&= \frac{1}{2}\{\|z_n-\widehat{u}\|^2+\|m_n-\widehat{u}\|^2-(\|z_n-m_n\|^2+\alpha^2\|h^*(I-T_{r_n}^G)hm_n\|^2\\
&\quad +2\alpha\langle z_n-m_n\ ,\ h^*(I-T_{r_n}^G)hm_n\rangle)\}.
\end{aligned}$$

So, we have

$$\begin{aligned}
\|z_n-\widehat{u}\|^2 &\leq \|m_n-\widehat{u}\|^2-\|z_n-m_n\|^2-2\alpha\langle z_n-m_n, h^*(I-T_{r_n}^G)hm_n\rangle\\
&\leq \|m_n-\widehat{u}\|^2-\|z_n-m_n\|^2+2\alpha\|z_n-m_n\|\|h^*(I-T_{r_n}^G)hm_n\|. \quad (3.23)
\end{aligned}$$

This implies that

$$\begin{aligned}
\|u_n-\widehat{u}\|^2 &\leq \beta_n\|m_n-\widehat{u}\|^2+(1-\beta_n)\|z_n-\widehat{u}\|^2\\
&\leq \beta_n\|m_n-\widehat{u}\|^2+(1-\beta_n)(\|m_n-\widehat{u}\|^2-\|z_n-m_n\|^2\\
&\quad +2\alpha\|z_n-m_n\|\|h^*(I-T_{r_n}^G)hm_n\|).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
(1-\beta_n)\|z_n-m_n\|^2 &\leq \|m_n-\widehat{u}\|^2-\|u_n-\widehat{u}\|^2\\
&\quad +2\alpha(1-\beta_n)\|z_n-m_n\|\|h^*(I-T_{r_n}^G)hm_n\|. \quad (3.24)
\end{aligned}$$

Utilizing (3.2-3.3), (3.22) and (C2), we have

$$\lim_{n\to\infty}\|z_n-m_n\|=0. \quad (3.25)$$

From (3.9) and (3.25), we have

$$\begin{aligned}
\lim_{n\to\infty}\|z_n-m_n+x_n-x_n\| &= 0\\
\lim_{n\to\infty}\|z_n-x_n-(m_n-x_n)\| &= 0\\
\lim_{n\to\infty}\|z_n-x_n\| &= 0. \quad (3.26)
\end{aligned}$$

Letting $n\to 0$ implies that $z_n\rightharpoonup\widehat{v}$. Next, we show that $\widehat{v}\in EP(F)$. Since, $z_n=T_{r_n}^F(I-\alpha h^*(I-T_{r_n}^G)h)m_n$, for any $y\in C$, we have

$$F(z_n,y)+\frac{1}{r_n}\langle y-z_n, z_n-x_n-\alpha h^*(I-T_{r_n}^G hm_n)\rangle\geq 0.$$

This implies that

$$F(z_n, y) + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle - \frac{1}{r_n}\langle y - z_n, \alpha h^*(I - T^G_{r_n} hm_n)\rangle \geq 0.$$

From the Assumption 2.3(A2), we have

$$\frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle - \frac{1}{r_n}\langle y - z_n, \alpha h^*(I - T^G_{r_n} hm_n)\rangle \geq -F(z_n, y) \geq F(\widehat{u}, z_n).$$

So, we have

$$\frac{1}{r_{n_i}}\langle y - z_{n_i}, z_{n_i} - x_{n_i}\rangle - \frac{1}{r_{n_i}}\langle y - z_{n_i}, \alpha h^*(I - T^G_{r_n} hm_{n_i})\rangle \geq F(y, z_{n_i}). \tag{3.27}$$

Utilizing (3.22) and (C3), we get that $z_{n_i} \rightharpoonup \widehat{v}$. Moreover, utilizing (3.34) and the Assumption 2.3(A2), we estimate

$$F(y, \widehat{v}) \leq 0, \quad \text{for all} \quad y \in C.$$

Let $y_t = ty + (1 - t)\widehat{v}$ for some $1 \geq t > 0$ and $y \in C$. Since $\widehat{v} \in C$, consequently, $y_t \in C$ and hence $F(y_t, \widehat{v}) \leq 0$. Using Assumption 2.3((A1) and (A4)), it follows that

$$\begin{aligned} 0 &= F(y_t, y_t) \\ &\leq tF(y_t, y) + (1 - t)F(y_t, \widehat{v}) \\ &\leq t(F(y_t, y)). \end{aligned}$$

This implies that

$$F(y_t, y) \geq 0, \quad \text{for all } y \in C.$$

Letting $t \to 0$ and by, Assumption 2.3 (A3), we get

$$F(\widehat{v}, y) \geq 0, \quad \text{for all } y \in C.$$

Thus, $\widehat{v} \in EP(F)$. Similarly, we can show that $\widehat{v} \in EP(G)$. Since h is a bounded linear operator, we have $hx_{n_i} \rightharpoonup h\widehat{v}$. It follow from (3.22) that

$$T^G_{r_{n_i}} hm_{n_i} \rightharpoonup h\widehat{v} \quad \text{as} \ \ i \to \infty. \tag{3.28}$$

Now, from Lemma 2.4 we have

$$G(T^G_{r_{n_i}} hm_{n_i}, y) + \frac{1}{r_{n_i}}\langle y - T^G_{r_{n_i}} hm_{n_i}, T^G_{r_{n_i}} hm_{n_i} - hm_{n_i}\rangle \geq 0,$$

for all $y \in C$. Since G is upper semicontinuous in the first argument and from (3.28), we have

$$G(h\widehat{v}, y) \geq 0,$$

for all $y \in C$. This implies that $h\widehat{v} \in EP(G)$. Therefore, $\widehat{v} \in SEP(F, G)$ and hence $\widehat{v} \in \Gamma$. This completes the proof.

∎

If we take $G = 0$ then split equilibrium problem goes to the classical equilibrium problem. So from (3.1), we get

Corollary 3.2. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a bifunctions satisfying (A1)-(A4) of Assumption 2.3. Let $A : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ be a maximally monotone operator and let $B : \mathcal{H}_1 \to \mathcal{H}_1$ be a monotone and ρ-Lipschitz operator for some $\rho > 0$. Assume that $\Gamma = (A+B)^{-1}(0) \cap \Omega \neq \emptyset$, where $\Omega := \{x^* \in C : x^* \in EP(F)\}$. For given $x_0, x_1 \in \mathcal{H}_1$ let the iterative sequence $\{x_n\}$,$\{m_n\}$,$\{u_n\}$,$\{v_n\}$ and $\{w_n\}$ are generated by*

$$\begin{cases} m_n = x_n + \gamma_n(x_n - x_{n-1}); \\ u_n = \beta_n m_n + (1-\beta_n) T^F_{r_n} m_n; \\ v_n = J_{\mu_n A}(I - \mu_n B) u_n; \\ w_n = v_n - \mu_n(Bv_n - Bu_n); \\ C_{n+1} = \{\widehat{u} \in C_n : \|w_n - \widehat{u}\|^2 \leq \|x_n - \widehat{u}\|^2 - 2\gamma_n \langle x_n - \widehat{u}, x_{n-1} - x_n \rangle + \gamma_n^2 \|x_{n-1} - x_n\|^2\}; \\ x_{n+1} = P_{C_n}(x_0). \end{cases} \tag{3.29}$$

Let a sequence $\{x_n\}_{n=0}^{\infty}$ in $\mathcal{H}$ be generated by (3.29), for each $n \geq 1$, where $\{\mu_n\} \subset (0, 2\rho)$, $\{\gamma_n\} \subset [0, \gamma]$, $\gamma \in [0, \frac{1}{2})$, $\{r_n\} \subset (0, \infty)$ and $\{\beta_n\}$ are sequence in $[0,1]$. Assume that the following conditions hold:

C1 $\sum_{n=1}^{\infty} \gamma_n \|x_n - x_{n-1}\| < \infty$;

C2 $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

C3 $\liminf_{n\to\infty} r_n > 0$;

C4 $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < \frac{2}{\rho} < 2\rho$.

Then the sequence $\{x_n\}$ generated by (3.29) strongly convergence to a point $\widehat{u} = P_\Gamma x_1$.

If we take $B := 0$ in (3.1), then we obtain the following corollary,

Corollary 3.3. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Let $F : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be two bifunctions satisfying Assumption 2.3 such that G is upper semicontinuous. Let $h : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator; let $A : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ be a maximally monotone operator. Assume that $\Gamma = (A)^{-1}(0) \cap \Omega \neq \emptyset$, where $\Omega := \{x^* \in C : x^* \in EP(F) \text{ and } hx^* \in EP(G)\}$. For given $x_0, x_1 \in \mathcal{H}_1$ let the iterative sequence $\{x_n\}$,$\{m_n\}$,$\{u_n\}$,$\{v_n\}$ and $\{w_n\}$ are generated by*

$$\begin{cases} m_n = x_n + \gamma_n(x_n - x_{n-1}); \\ u_n = \beta_n m_n + (1-\beta_n) T^F_{r_n}(I - \alpha h^*(I - T^G_{r_n})h) m_n; \\ v_n = J_{\mu_n A} u_n; \\ w_n = v_n; \\ C_{n+1} = \{\widehat{u} \in C_n : \|w_n - \widehat{u}\|^2 \leq \|x_n - \widehat{u}\|^2 - 2\gamma_n \langle x_n - \widehat{u}, x_{n-1} - x_n \rangle + \gamma_n^2 \|x_{n-1} - x_n\|^2\}; \\ x_{n+1} = P_{C_n}(x_0). \end{cases} \tag{3.30}$$

Let a sequence $\{x_n\}_{n=0}^{\infty}$ in $\mathcal{H}$ be generated by (3.30), for each $n \geq 1$, where $\{\mu_n\} \subset (0, 2\rho)$, $\{\gamma_n\} \subset [0, \gamma]$, $\gamma \in [0, \frac{1}{2})$, $\{r_n\} \subset (0, \infty)$, with $\alpha \in (0, \frac{1}{L})$ such that L is the spectral radius of h^*h and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$. Assume that the following conditions hold:

C1 $\sum_{n=1}^{\infty} \gamma_n \|x_n - x_{n-1}\| < \infty$;

C2 $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

C3 $\liminf_{n\to\infty} r_n > 0$;

C4 $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < 2\rho$.

Then the sequence $\{x_n\}$ generated by (3.30) strongly convergence to a point $\widehat{u} = P_\Gamma x_1$.

Remark 3.4. we remark here that the condition C1, it is easily applicable in numerical calculation since the valued of $\|x_n - x_{n-1}\|$ is known before choosing γ_n. At here, the parameter γ_n can be taken as $0 \leq \gamma_n \leq \widehat{\gamma_n}$,

$$\widehat{\gamma_n} = \begin{cases} \min\{\frac{w_n}{\|x_n - x_{n-1}\|}, \gamma\} & \textit{if} \quad x_n \neq x_{n-1}; \\ \gamma & \text{othrwise,} \end{cases}$$

where $\{w_n\}$ is a positive sequence such that $\sum_{n=1}^{\infty} w_n < \infty$ and $\gamma \in [0, 1)$.

4. Applications

In this section, we illustrate the theoretical results which we already obtained in previous section.

Convex Minimization Problem:

Let $\mathfrak{f} : \mathcal{H} \to \mathbb{R}$ and $\mathfrak{g} : \mathcal{H} \to \mathbb{R}$ be two convex, proper and lower semicontinuous functions such that a function $\mathfrak{f}$ and its differentiable with L-Lipschitz continuous gradient and another one function $\mathfrak{g}$ which is sub-differential and it is easily calculated. Assume that ω is the set of solutions of problem (2.1) and $\omega \neq 0$. In theorem 3.3, set that $B := \nabla\mathfrak{f}$ and $A := \partial\mathfrak{g}$. Then, we compute the following theorem for solving (2.1) in strong convergence with inertial and split equilibrium problem form.

Corollary 4.1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Let $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ be two bifunctions satisfying (A1)-(A4) of Assumption 2.3 such that G is upper semicontinuous. Let $h : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Let $A : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ be a maximally monotone operator and let $B : \mathcal{H}_1 \to \mathcal{H}_1$ be a monotone and ρ-Lipschitz operator for some $\rho > 0$. Let $\mathfrak{f}, \mathfrak{g} : \mathcal{H}_1 \to \mathbb{R}$ be two convex, proper and lower semicontinuous functions, such that a function $\mathfrak{f}$ which is differentiable with ρ-Lipschitz continuous gradient and another function $\mathfrak{g}$ which is sub-differential and it is easily calculated. Assume that ω is the set of solutions of problem (2.1) and $\omega \neq 0$. Let γ_n be a bounded real sequence and $\mu \in (0, \frac{2}{\rho})$. Assume that $\Gamma = \omega \cap \Omega \neq \emptyset$, where $\Omega := \{x^* \in C : x^* \in EP(F) \quad \text{and} \quad hx^* \in EP(G)\}$. For given $x_0, x_1 \in \mathcal{H}_1$ let the iterative sequence $\{x_n\}$,$\{m_n\}$,$\{u_n\}$,$\{v_n\}$ and $\{w_n\}$ are generated by*

$$\begin{cases} m_n = x_n + \gamma_n(x_n - x_{n-1}); \\ u_n = \beta_n m_n + (1-\beta_n)T^F_{r_n}(I - \alpha h^*(I - T^G_{r_n})h)m_n; \\ v_n = prox_{\mu_n \mathfrak{g}}(I - \mu_n \nabla\mathfrak{f})u_n; \\ w_n = v_n - \mu_n(\nabla\mathfrak{f}v_n - \nabla\mathfrak{f}u_n); \\ C_{n+1} = \{\widehat{u} \in C_n : \|w_n - \widehat{u}\|^2 \leq \|x_n - \widehat{u}\|^2 - 2\gamma_n\langle x_n - \widehat{u}, x_{n-1} - x_n\rangle + \gamma_n^2\|x_{n-1} - x_n\|^2\}; \\ x_{n+1} = P_{C_n}(x_0). \end{cases} \tag{4.1}$$

Let a sequence $\{x_n\}_{n=0}^{\infty}$ in $\mathcal{H}$ be generated by (4.1), for each $n \geq 1$, where$\{\mu_n\} \subset (0, \frac{2}{\rho})$, $\{\gamma_n\} \subset [0, \gamma]$, $\gamma \in [0, 1)$, $\{r_n\} \subset (0, \infty)$, with $\alpha \in (0, \frac{1}{L})$ such that L is the spectral radius of h^*h and $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0, 1]$. Assume that the following conditions hold:

C1 $\sum_{n=1}^{\infty} \gamma_n\|x_n - x_{n-1}\| < \infty$;

C2 $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

C3 $\liminf_{n\to\infty} r_n > 0$;

C4 $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < \frac{2}{\rho} < 2\rho$.

Then the sequence $\{x_n\}$ generated by Theorem (4.1) strongly convergence to a point $\widehat{u} = P_\Gamma x_1$.

Split Feasibility Problem: Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and $h : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Let C and Q be closed, convex and nonempty subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The split feasibility problem (SFP) is the problem to find $\widehat{x} \in C$ such that $S\widehat{x} \in Q$. We represent the solution sets by $\omega := C \cap h^{-1}(Q) = \{\widehat{y} \in C : h\widehat{y} \in Q\}$. Censor and Elfving [15] introduced first time it, to solve inverse problems and their application to medical image reconstruction and radiation therapy and modeling and simulation in a finite dimensional Hilbert space. Recall C is the function

$$b_C(\widehat{x}) := \begin{cases} 0, & \widehat{x} \in C; \\ \infty, & otherwise. \end{cases}$$

The proximal mapping of b_C is the metric projection on C,

$$\begin{aligned} prox_{b_C} &= \arg\min_{\widehat{p}\in C} \|\widehat{p} - \widehat{x}\| \\ &= P_C(\widehat{x}). \end{aligned}$$

Let $h : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator and h^* the adjoint of h. Let P_Q be the projection of $\mathcal{H}_2$ onto a nonempty, convex and closed subset Q. Take: $f(\widehat{x}) = \frac{1}{2}\|h\widehat{x} - P_Q h\widehat{x}\|^2$ and $g(\widehat{x}) = b_C(\widehat{x})$. Then, we compute the split feasibility problem from following theorem in strong convergence with inertial and split equilibrium problem form.

Corollary 4.2. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Let $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ be two bifunctions satisfying (A1)-(A4) of Assumption 2.3 such that G is upper semicontinuous. Let $h : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Let $A : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ be a maximally monotone operator and let $B : \mathcal{H}_1 \to \mathcal{H}_1$ be a monotone and ρ-Lipschitz operator for some $\rho > 0$. Assume that ω is the set of solutions of problem (2.1) and $\omega \neq 0$. Let γ_n be a bounded real sequence and $\mu \in (0, \frac{2}{\|h\|^2})$. Assume that $\Gamma = \omega \cap \Omega \neq \emptyset$, where $\Omega := \{x^* \in C : x^* \in EP(F) \quad and \quad hx^* \in EP(G)\}$. For given $x_0, x_1 \in \mathcal{H}_1$ let the iterative sequence $\{x_n\}$,$\{m_n\}$,$\{u_n\}$,$\{v_n\}$ and $\{w_n\}$ are generated by*

$$\begin{cases} m_n = x_n + \gamma_n(x_n - x_{n-1}); \\ u_n = \beta_n m_n + (1-\beta_n) T^F_{r_n}(I - \alpha h^*(I - T^G_{r_n})h)m_n; \\ v_n = P_C(I - \mu_n h^*(I - P_Q)h)u_n; \\ w_n = v_n - \mu_n(h^*(I - P_Q)hv_n - h^*(I - P_Q)hu_n); \\ C_{n+1} = \{\widehat{u} \in C_n : \|w_n - \widehat{u}\|^2 \leq \|x_n - \widehat{u}\|^2 - 2\gamma_n\langle x_n - \widehat{u}, x_{n-1} - x_n\rangle + \gamma_n^2\|x_{n-1} - x_n\|^2\}; \\ x_{n+1} = P_{C_n}(x_0). \end{cases} \tag{4.2}$$

Let a sequence $\{x_n\}_{n=0}^\infty$ in $\mathcal{H}$ be generated by (4.2), for each $n \geq 1$, where$\{\mu_n\} \subset (0, \frac{2}{\|h\|^2})$, $\{\gamma_n\} \subset [0, \gamma]$, $\gamma \in [0, 1)$, $\{r_n\} \subset (0, \infty)$, with $\alpha \in (0, \frac{1}{L})$ such that L is the spectral radius of h^*h and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$. Assume that the following conditions hold:

C1 $\sum_{n=1}^{\infty}\gamma_n\|x_n - x_{n-1}\| < \infty$;

C2 $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

C3 $\liminf_{n\to\infty} r_n > 0$;

C4 $0 < \liminf_{n\to\infty}\mu_n \leq \limsup_{n\to\infty}\mu_n < \frac{2}{\|h\|}$.

Then the sequence $\{x_n\}$ generated by Theorem (4.2) strongly convergence to a point $\widehat{u} = P_\Gamma x_1$.

5. Example and Numerical Results

This section shows effectiveness to our algorithm by following given examples and numerical results.

Example 5.1. Let $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y\rangle = xy$, for all $x, y \in \mathbb{R}$ and induced usual norm $|\cdot|$. Let $F : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is defined as $F(x, y) = 2x(y - x)$ where $x, y \in F$ and let $G : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is defined as $G(u, v) = u(v - u)$ where $u, v \in G$. For $r > 0$, we define three mappings $h, A, B : \mathbb{R} \to \mathbb{R}$ are defined as $h(x) = 3x$, $Ax = 4x$ and $Bx = 3x$, respectively. For all $x = x_0, x_1 \in \mathbb{R}$ and B be a monotone and ρ-Lipschitz operator for some $\rho > 0$ and A is maximal monotone. Then there exist unique sequences $\{x_n\}$,$\{m_n\}$,$\{u_n\}$,$\{v_n\}$ and $\{w_n\}$ are generated by iterative method in theorem (3.1). Choose $\alpha = 0.5$,$\beta = \frac{n}{100n-1}$, $r = \frac{n}{100n-1}$, $L = 3$ and μ=0.004.

Since $\gamma_n = \begin{cases} \min\{\frac{1}{n^2\|x_n - x_{n-1}\|}, 0.5\} & if \quad x_n \neq x_{n-1}; \\ 0.5 & \text{otherwise,} \end{cases}$

then $\{x_n\}$ conveges strongly.

It is easy to prove that the bifunction F and G satisfy the A_1-A_2 and G is upper semicontinuous. h is bounded linear operator on $\mathbb{R}$ with adjoint opertator h^* and $\|h\| = \|h^*\| = 3$. Moreover, $Sol(EP(F) = \{0\}$, $Sol(EP(G)) = \{0\}$. Hence $\Gamma = (A + B)^{-1}(0) \cap \Omega = 0$. Now, we solved this numerical example in six step,

Step 1. Find $z \in Q$ such that $G(z, y) + \frac{1}{r}\langle y - z, z - hx\rangle \geq 0$ for all $y \in Q$.

$$\begin{aligned} G(z, y) + \frac{1}{r}\langle y - z, z - hx\rangle \geq 0 \quad &\Leftrightarrow \quad z(y - z) + \frac{1}{r}\langle y - z, z - hx\rangle \geq 0, \\ &\Leftrightarrow \quad rz(y - z) + (y - z)(z - hx) \geq 0, \\ &\Leftrightarrow \quad (y - z)((1 + r)z - hx) \geq 0 \end{aligned}$$

for all $y \in Q$. Thus, By Lemma 2.4(2), we know that $T_r^G hx$ is single-valued for each $x \in C$. Hence $z = \frac{hx}{1+r}$.

Step 2. Find $m \in C$ such that $m = x - \alpha h^*(I - T_r^G)hx$. From Step 1, we get,

$$\begin{aligned} m = x - \alpha h^*(I - T_r^G)hx \quad &= \quad x - \alpha h^*(I - T_r^G)hx, \\ &= \quad x - \alpha(3x - \frac{3(hx)}{1 + r}), \\ &= \quad (1 - 3\alpha)x + \frac{3\alpha}{1 + r}(hx). \end{aligned}$$

Step 3.
Find $u \in C$ such that $F(u,v) + \frac{1}{r}\langle v-u, u-m\rangle \geq 0$ for all $v \in C$. From Step Two, we have

$$\begin{aligned} F(u,v) + \frac{1}{r}\langle v-u, u-m\rangle \geq 0 \quad &\leftrightarrow \quad (2u)(v-u) + \frac{1}{r}\langle v-u, u-m\rangle \geq 0, \\ &\leftrightarrow \quad r(2u)(v-u) + (v-u)(u-m) \geq 0, \\ &\leftrightarrow \quad (v-u)((1+2r)u-m) \geq 0 \end{aligned}$$

for all $v \in C$. Similarly, by Lemma 2.4(2), we obtain $u = \frac{m}{1+2r} = \frac{(1-3\alpha)x}{1+2r} + \frac{3\alpha hx}{(1+r)(1+2r)}$.

Step 4.
Formulations for the sequences.

$$\begin{cases} x_0 = x \in \mathbb{R}; \\ m_n = x_n + \gamma_n(x_n - x_{n-1}); \\ u_n = \frac{n}{100n+1}m_n + (1 - \frac{n}{100n+1})(\frac{(1-3\alpha)x_n}{1+2r} + \frac{3\alpha hx_n}{(1+r)(1+2r)})m_n, \\ v_n = (\frac{1-3s}{1+4s}x_n - \frac{s}{1+4s}3x_n)u_n, \\ w_n = v_n - 0.004(3v_n - 3u_n); \end{cases}$$

Step 5.
Find $C_{n+1} = \{\widehat{u} \in C_n : \|w_n - \widehat{u}\|^2 \leq \|x_n - \widehat{u}\|^2 - 2\gamma_n\langle x_n - \widehat{u}, x_{n-1} - x_n\rangle + \gamma_n^2\|x_{n-1} - x_n\|^2\}$.
Since $\|w_n - \widehat{u}\|^2 \leq \|x_n - \widehat{u}\|^2 - 2\gamma_n\langle x_n - \widehat{u}, x_{n-1} - x_n\rangle + \gamma_n^2\|x_{n-1} - x_n\|^2\}$ we have

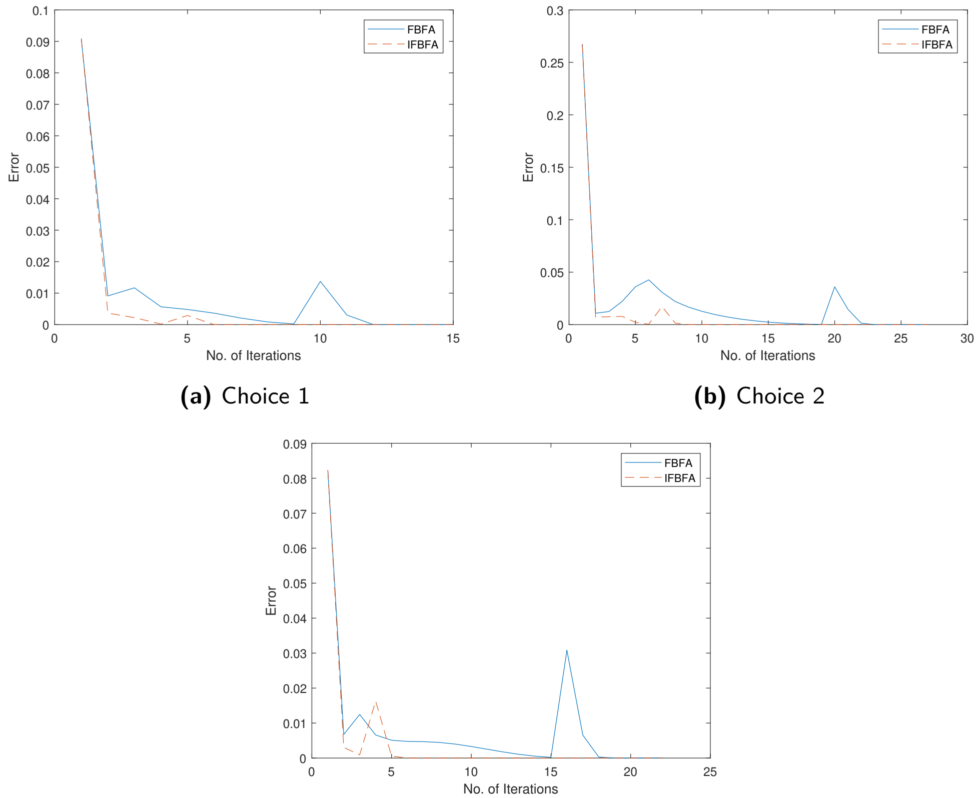
$$\frac{w_n + x_n}{2} \leq \widehat{u}.$$

Step 6.
Compute the numerical results of $x_{n+1} = P_{C_{n+1}}x_1$.
We provide a numerical test of a comparison between our inertial forward-backward-forward method defined in Theorem 3.2 (i.e $\gamma_n \neq 0$)and standard forward-backward-forward method (i.e $\gamma_n = 0$). The stoping criteria is defined as $E_n = \|x_{n+1} - x_n\| < 10^{-6}$. The different choices of x_0 and x_1 are giving as following:

Table 1. Numerical results for Example 5.1

	No. of Iter.		**CPU(Sec)**	
	$\gamma_n = 0$	$\gamma_n \neq 0$	$\gamma_n = 0$	$\gamma_n \neq 0$
Choice 1. $x_0 = (2)$ and $x_1 = (-2)$	18	11	0.044675	0.041939
Choice 2. $x_0 = (10)$ and $x_1 = (2)$	29	13	0.051017	0.042856
Choice 3. $x_0 = (1.5)$ and $x_1 = (2.5)$	24	10	0.043589	0.037694

The error plotting E_n of $\gamma_n \neq 0$ and $\gamma_n = 0$ for each choices in Table 1. is shown in figure 1-3, respectively,



(a) Choice 1

(b) Choice 2

(c) Choice 3

Fig. 1. Evaluation of iterations for IFBFA and FBFA in Choice 3 of Example 5.1

Conclusion. The main aim of this paper is to propose an iterative algorithm to find an element for solving a class of split equilibrium problem and inclusion problem in Hilbert spaces. We introduce a modified inertial forward-backward-forward splitting algorithm and its convergence theorem for the split equilibrium problem and Inclusion problem in Hilbert spaces. We also proved there convergence and designed the algorithms by combining the forward-backward-forward splitting method and the shrinking projection method. Some applications and numerical example and computational results are implemented for bifunctions, which are generalized from the split equilibrium problem to illustrate the convergence which are presented in this paper.

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