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A Spectral Conjugate Gradient-like Method for Convex Constrained Nonlinear Monotone Equations and Signal Recovery

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ABSTRACT

Many real-world problems can be formulated as systems of nonlinear equations. Thus, finding their solutions is of paramount importance. Traditional approaches such as Newton and quasi-Newton methods for solving these systems require computing Jacobian matrix or an approximation to it at every iteration, which is very expensive especially when the dimension of the systems is large. In this work, we propose a derivative-free algorithm for solving these systems. The proposed algorithm is a combination of the popular conjugate gradient method for unconstrained optimization problems and the projection method. We prove the global convergence of the proposed algorithm under Lipschitz continuity and monotonicity assumptions on the underlying mapping. We perform numerical experiments on some test problems, and the proposed algorithm proves to be more efficient in comparison with some existing works. Finally, we give an application of the proposed algorithm in signal recovery.

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1. Introduction

In recent years, spectral and conjugate gradient projection based methods have received much attention in solving convex constrained systems of nonlinear monotone equations given as:

$$J(x) = 0, \quad \text{subject to} \quad x \in \Lambda \subseteq \mathbb{R}^n, \tag{1.1}$$

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where Λ is nonempty, closed and convex, and $J : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and monotone. Monotone means

$$\langle J(x) - J(y),\ x - y \rangle \geq 0 \quad \text{for all} \quad x, y \in \mathbb{R}^n.$$

Problem (1.1) arises in a number of applications from engineering and other branches of sciences. For example, in economic equilibrium problems [11], power flow equations [28] and chemical equilibrium systems [22]. Moreover, algorithms for solving systems of nonlinear monotone equations are used in signal and image recovery, (see [14, 29, 30, 3, 1, 4, 17, 2, 6]).

Classical methods for solving (1.1) include Newton and quasi-Newton methods which have fast convergence from good initial guess. However, the main problem associated with these methods include solving linear system using a Jacobian matrix or its approximation at every iteration. As a result, these methods are not suitable to handle large scale problems. On the other hand, spectral and conjugate gradient methods proposed for unconstrained optimization problems do not require any Jacobian matrix or its approximation and are simple to implement. These important properties make them suitable for solving large scale optimization problems. Motivated by these nice properties, researchers extended these methods to solve system of nonlinear equations [9, 18, 19]. Conjugate gradient method produces sequence of iterations using the formula:

$$x_{k+1} = x_k + \alpha_k d_k, \tag{1.2}$$

where α_k is a step size, and d_k is a search direction. To solve (1.1), the definition of the search direction is given as:

$$d_k = \begin{cases} -J(x_k), & \text{if } k = 0, \\ -J(x_k) + \beta_k d_{k-1}, & \text{if } k \geq 1, \end{cases} \tag{1.3}$$

where the parameter β_k is a scalar known as the conjugate gradient parameter.

Following the success of the projection technique proposed by Solodov and Svaiter [26], many conjugate gradient projection based methods have been proposed to handle large scale systems of nonlinear equations. For example, Cheng [8] combined the projection technique in [26] and the Polak–Ribière–Polyak (PRP) method [24, 25] to propose a conjugate gradient projection based method for solving nonlinear monotone equations. They proved the global convergence of the method under monotonicity and Lipschitz continuity assumption of the mapping considered. The numerical results presented proved that their method is promising in solving large scale problems. In the work of Xiao and Zhou [30], based on the projection technique in [26], an extension of the popular descent conjugate gradient method (CG_Descent) [15] is proposed for solving convex constrained monotone nonlinear equations. Liu and Li [21] presented another extension of the CG_Descent method to solve convex constrained nonlinear monotone equations. They showed that their proposed method is globally convergent and has some advantages numerically when compared with the method proposed in [30]. Liu and Feng [20] propose a spectral conjugate gradient method for solving convex constrained monotone nonlinear equations. Their work is also a combination of the projection technique [26] and the popular Dai-Yuan conjugate gradient parameter [10]. The numerical results proved that their method is more efficient than the ones proposed in [31] and [21].

Inspired by the above contributions, and the success of the projection technique in [26], we propose a spectral conjugate gradient projection based method for solving (1.1). Among the advantages of the proposed method is that it inherits the low storage requirement property of the spectral conjugate gradient method, and thus, it is suitable to solve large scale nonlinear monotone equations. It is derivative-free, and the global convergence is established without any differentiability assumption. Moreover, we perform numerical experiments on some test problems to depict the efficiency of the proposed algorithm in comparison with the ones proposed in [20] and [33]. Additionally, we apply the proposed algorithm in signal recovery problems, and the quality of the recovered signal proves that the proposed algorithm is more efficient than some existing methods.

The remaining part of this paper is organized as follows. In the next section, we introduce the proposed algorithm, some important definitions and prove global convergence. Then, in section 3, we present some numerical experiments of the proposed algorithm and compare it performance with two existing ones. This is followed by a section where we show the application of the proposed algorithm in signal recovery. In the last section, we give the conclusion of the work.

2. Algorithm and Convergence Analysis

We begin this section by defining the projection operator as follows:

Definition 2.1. Let $\Lambda \subset \mathbb{R}^n$ be a nonempty closed convex set. Then for any $x \in \mathbb{R}^n$, its projection onto Λ, denoted by $P_\Lambda(x)$, is defined by

$$P_\Lambda(x) = \arg\min\{\|x - y\| \ : y \in \Lambda\}.$$

The projection operator P_Λ has the properties

$$\|P_\Lambda(x) - P_\Lambda(y)\| \leq \|x - y\|, \quad \forall x, y \in \mathbb{R}^n, \tag{2.1}$$

and

$$\|P_\Lambda(x) - y\| \leq \|x - y\|, \quad \forall y \in \Lambda. \tag{2.2}$$

In this work, we define a new search direction as follows:

$$d_k = \begin{cases} -J(x_k), & \text{if } k = 0, \\ -\tau_k J(x_k) + \frac{\|J(x_k)\|^2}{-J(x_{k-1})^T d_{k-1}} s_{k-1}, & \text{if } k \geq 1, \end{cases} \tag{2.3}$$

where $s_{k-1} = \alpha_{k-1} d_{k-1}$, and the parameter τ_k is obtained such that the direction (2.3) satisfies

$$J(x_k)^T d_k \leq -c\|J(x_k)\|^2, \tag{2.4}$$

which is an important property in establishing the global convergence.

Observe that from (2.3), when $k = 0$, $J(x_k)^T d_k = -\|J(x_k)\|^2$, thus, (2.4) is satisfied with $c = 1$. However, when $k \geq 1$,

$$\begin{aligned} J(x_k)^T d_k &= -\tau_k\|J(x_k)\|^2 - \frac{\|J(x_k)\|^2 J(x_k)^T s_{k-1}}{J(x_{k-1})^T d_{k-1}} \\ &= -(\tau_k + \frac{J(x_k)^T s_{k-1}}{J(x_{k-1})^T d_{k-1}})\|J(x_k)\|^2 \end{aligned} \tag{2.5}$$

To satisfy (2.4), we only need $\tau_k + \frac{J(x_k)^T s_{k-1}}{J(x_{k-1})^T d_{k-1}} \geq c$, $c > 0$. That is, $\tau_k \geq c - \frac{J(x_k)^T s_{k-1}}{J(x_{k-1})^T d_{k-1}}$. In this work, we choose

$$\tau_k = c - \frac{J(x_k)^T s_{k-1}}{J(x_{k-1})^T d_{k-1}}. \tag{2.6}$$

Thus, it is not difficult to see that $\forall k \geq 1$, multiplying (2.3) by $J(x_k)^T$ and substituting $\tau_k = c - \frac{J(x_k)^T s_{k-1}}{J(x_{k-1})^T d_{k-1}}$ gives

$$J(x_k)^T d_k = -c\|J(x_k)\|^2. \tag{2.7}$$

Furthermore, taking absolute value from (2.7), we get $|J(x_k)^T d_k| = c\|J(x_k)\|^2, \ \forall k$ and consequently,

$$|J(x_{k-1})^T d_{k-1}| = c\|J(x_{k-1})\|^2. \tag{2.8}$$

We now give the steps of our proposed algorithm as follows:

Algorithm 1: Spectral Conjugate Gradient Projection Method (SCD)

Input : Choose initial point $x_0 \in \Lambda$, $\gamma \in (0,2), \sigma \in (0,1)$, $\kappa \in (0,1]$, $c > 0$, $Tol > 0$ and $\beta \in (0,1)$. Set $k := 0$

Step 1: If $\|J(x_k)\| \leq Tol$, stop, otherwise go to **Step 2**.

Step 2: Compute d_k using equation (2.3).

Step 3: Compute the step size $\alpha_k = \max\{\kappa\beta^i \ : i = 0,1,2,\cdots\}$ such that

$$-J(x_k + \kappa\beta^i d_k)^T d_k \geq \sigma\kappa\beta^i \|d_k\|^2. \tag{2.9}$$

Step 4: Set $z_k = x_k + \alpha_k d_k$. If $z_k \in \Lambda$ and $\|J(z_k)\| = 0$, stop. Else compute

$$x_{k+1} = P_\Lambda[x_k - \gamma\zeta_k J(z_k)],$$

where

$$\zeta_k = \frac{J(z_k)^T (x_k - z_k)}{\|J(z_k)\|^2}.$$

Step 5: Let $k = k+1$ and go to **Step 1**.

In order to establish the global convergence of the proposed algorithm, we assumed the following:

(Q_1) The mapping J is monotone.

(Q_2) The mapping J is Lipschitz continuous, that is there exists a positive constant L such that

$$\|J(x) - J(y)\| \leq L\|x - y\|, \ \forall x, y \in \mathbb{R}^n.$$

(Q_3) The solution set of (1.1), denoted by Λ, is nonempty.

Lemma 2.2. *Suppose that assumptions (Q_1)-(Q_3) hold, then the sequences $\{x_k\}$ and $\{z_k\}$ generated by Algorithm 1 are bounded. Moreover, we have*

$$\lim_{k\to\infty} \|x_k - z_k\| = 0, \tag{2.10}$$

and

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0. \tag{2.11}$$

Proof. Let $\bar{x}$ be the solution of (1.1). Then, by monotonicity of the mapping J, we get

$$\begin{aligned} \langle J(z_k),\ x_k - \bar{x}\rangle &= \langle J(z_k),\ x_k - z_k + z_k - \bar{x}\rangle \\ &= \langle J(z_k),\ x_k - z_k\rangle + \langle J(z_k) - J(\bar{x}),\ z_k - \bar{x}\rangle \\ &\geq \langle J(z_k),\ x_k - z_k\rangle. \end{aligned} \tag{2.12}$$

Using x_{k+1} definition from **Step 4**, equation (2.2) and (2.12) we obtain

$$\begin{aligned} \|x_{k+1} - \bar{x}\|^2 &= \|P_\Lambda[x_k - \gamma\zeta_k J(z_k)] - \bar{x}\|^2 \leq \|x_k - \gamma\zeta_k J(z_k) - \bar{x}\|^2 \\ &= \|x_k - \bar{x}\|^2 - 2\gamma\zeta_k J(z_k)^T(x_k - \bar{x}) + \gamma^2\zeta_k^2\|J(z_k)\|^2 \\ &= \|x_k - \bar{x}\|^2 - 2\gamma\frac{J(z_k)^T(x_k - z_k)}{\|J(z_k)\|^2}J(z_k)^T(x_k - \bar{x}) + \gamma^2\left(\frac{J(z_k)^T(x_k - z_k)}{\|J(z_k)\|}\right)^2 \\ &\leq \|x_k - \bar{x}\|^2 - 2\gamma\frac{J(z_k)^T(x_k - z_k)}{\|J(z_k)\|^2}J(z_k)^T(x_k - z_k) + \gamma^2\left(\frac{J(z_k)^T(x_k - z_k)}{\|J(z_k)\|}\right)^2 \\ &= \|x_k - \bar{x}\|^2 - \gamma(2-\gamma)\left(\frac{J(z_k)^T(x_k - z_k)}{\|J(z_k)\|}\right)^2 \\ &\leq \|x_k - \bar{x}\|^2 - \gamma(2-\gamma)\frac{\sigma^2\|x_k - z_k\|^4}{\|J(z_k)\|^2}. \end{aligned} \tag{2.13}$$

This shows that the sequence $\{\|x_k - \bar{x}\|\}$ is a decreasing sequence, and hence $\{x_k\}$ is bounded. In addition, combining this with continuity of J, we can find $n_1 > 0$ such that

$$\|J(x_k)\| \leq n_1. \tag{2.14}$$

Since $(J(x_k) - J(z_k))^T(x_k - z_k) \geq 0$, using Cauchy-Schwarz inequality, we obtain

$$\|J(x_k)\|\|x_k - z_k\| \geq J(x_k)^T(x_k - z_k) \geq J(z_k)^T(x_k - z_k) \geq \sigma\|x_k - z_k\|^2,$$

where the above inequality follows from the definition of the line search and setting $z_k = x_k + \alpha_k d_k$ which gives

$$J(z_k)^T(x_k - z_k) = -\alpha_k J(z_k)d_k \geq \sigma\alpha_k^2\|d_k\|^2 = \sigma\|x_k - z_k\|^2.$$

Therefore,

$$\sigma\|x_k - z_k\| \leq \|J(x_k)\| \leq n_1,$$

showing that $\{z_k\}$ is bounded.

Again, using the continuity of J, we can find another constant $n_2 > 0$ such that $\|J(z_k)\| \leq n_2$ for all $k \geq 0$ this, together with (2.13) give us

$$\gamma(2-\gamma)\frac{\sigma^2}{n_2^2}\|x_k - z_k\|^4 \leq \|x_k - \bar{x}\|^2 - \|x_{k+1} - \bar{x}\|^2, \tag{2.15}$$

adding (2.15) for $k = 0, 1, 2, \cdots$, we have

$$\gamma(2-\gamma)\frac{\sigma^2}{n_2^2}\sum_{k=0}^{\infty}\|x_k - z_k\|^4 \leq \sum_{k=0}^{\infty}(\|x_k - \bar{x}\|^2 - \|x_{k+1} - \bar{x}\|^2) \leq \|x_0 - \bar{x}\|^2, \tag{2.16}$$

which implies

$$\lim_{k\to\infty}\|x_k - z_k\| = 0.$$

This and the definition of z_k implies

$$\lim_{k\to\infty}\alpha_k\|d_k\| = 0. \tag{2.17}$$

From the definition of projection operator, we get

$$\begin{aligned}
\lim_{k\to\infty}\|x_{k+1} - x_k\| &= \lim_{k\to\infty}\|P_\Lambda[x_k - \gamma\zeta_k J(z_k)] - x_k\| \\
&\leq \lim_{k\to\infty}\|x_k - \gamma\zeta_k J(z_k) - x_k\| \\
&\leq \gamma\lim_{k\to\infty}\|\zeta_k J(z_k)\| \\
&\leq \gamma\lim_{k\to\infty}\|x_k - z_k\| \\
&= 0.
\end{aligned} \tag{2.18}$$

∎

Lemma 2.3. *Suppose assumptions (Q_1)-(Q_3) hold, and the sequences $\{x_k\}$ and $\{z_k\}$ are generated by Algorithm 1. Then*

$$\alpha_k \geq \max\left\{\kappa, \frac{c\beta\|J(x_k)\|^2}{(L+\sigma)\|d_k\|^2}\right\}. \tag{2.19}$$

Proof. From the line search (2.9), if $\alpha_k \neq \kappa$, then $\alpha_k' = \alpha_k\beta^{-1}$ does not satisfy (2.9), that is,

$$-J(x_k + \alpha_k' d_k)^T d_k < \sigma\alpha_k'\|d_k\|^2.$$

Using (2.7) and assumption (Q_2), we have

$$\begin{aligned}
c\|J(x_k)\|^2 &= -J(x_k)^T d_k \\
&= (J(x_k + \alpha_k' d_k) - J(x_k))^T d_k - J(x_k + \alpha_k' d_k)^T d_k \\
&\leq \alpha_k'(L+\sigma)\|d_k\|^2.
\end{aligned}$$

Replacing $\alpha_k' = \alpha_k\beta^{-1}$ and solving for α_k gives the required result. ∎

Theorem 2.4. *Suppose that assumptions (Q_1)-(Q_3) hold, and let the sequence $\{x_k\}$ be generated by Algorithm 1, then*

$$\liminf_{k\to\infty}\|J(x_k)\| = 0. \tag{2.20}$$

Proof. Suppose by contradiction the relation (2.20) is not satisfied, then there exist a positive constant r_1 such that $\forall k \geq 0$,

$$\|J(x_k)\| \geq r_1. \tag{2.21}$$

From (2.7) and (2.21), we have that $\forall k \geq 0$,

$$\|d_k\| \geq cr_1. \tag{2.22}$$

From (2.3), (2.6), (2.8), (2.14) and (2.21), we have

$$\begin{aligned} \|d_k\| &= \left\| -\left(c - \frac{J(x_k)^T s_{k-1}}{J(x_{k-1})^T d_{k-1}}\right) J(x_k) - \frac{\|J(x_k)\|^2 s_{k-1}}{J(x_{k-1})^T d_{k-1}} \right\| \\ &\leq c\|J(x_k)\| + 2\frac{\|J(x_k)\|^2\|s_{k-1}\|}{c\|J(x_{k-1})\|^2} \\ &\leq cn_1 + \frac{2n_1^2\alpha_{k-1}\|d_{k-1}\|}{cr_1^2}. \end{aligned} \tag{2.23}$$

Equation (2.17) implies $\forall\epsilon_0 > 0$ there exist k_0 such that $\alpha_{k-1}\|d_{k-1}\| \leq \epsilon_0 \ \forall k > k_0$. Therefore, choosing $\epsilon_0 = r_1^2$ and $N = \max\{\|d_0\|, \|d_1\|, \|d_2\|, \cdots, \|d_{k_0}\|, M_1\}$ where $M_1 = cn_1 + \frac{2n_1^2}{c}$, we have $\|d_k\| \leq N$. Multiplying both sides of (2.19) with $\|d_k\|$ we get

$$\begin{aligned} \alpha_k\|d_k\| &\geq \max\left\{\kappa, \frac{c\beta\|J(x_k))\|^2}{(L+\sigma)\|d_k\|^2}\right\}\|d_k\| \\ &\geq \max\left\{\kappa cr_1\ , \ \frac{c\beta r_1^2}{(L+\sigma)N}\right\}. \end{aligned}$$

Taking the limit as $k \to \infty$ on both sides, we get

$$\lim_{k\to\infty} \alpha_k\|d_k\| > 0. \tag{2.24}$$

This contradicts (2.17). Hence,

$$\liminf_{k\to\infty} \|J(x_k)\| = 0. \tag{2.25}$$

∎

3. Numerical Experiments

In this section, we present the numerical experiments of our proposed SCD algorithm in comparison with two existing algorithms, specifically, the PDY algorithm proposed by Liu and Feng [20], and the algorithm (which we called LLY for simplicity) proposed by Zheng et al. [33]. All codes are written on Matlab R2019b and are run on a PC of corei3-4005U processor, 4 GB RAM and 1.70 GHZ CPU.

In PDY and LLY algorithms, we fixed the parameters as reported in the respective papers [20] and [33]. However, in our SCD algorithm, we choose $\beta = 0.6$, $\sigma = 0.0001$, $\kappa = 1$, $c = 1$ and $\gamma = 1.8$. We perform the experiments on seven test problems with eight initial points. These problems are tested on five different dimensions: $n = 1000$, $n = 5000$, $n = 10000$, $n = 50000$ and $n = 100000$. We used $\|J(x_k)\| \leq 10^{-5}$ as a stopping criteria.We now state the test problems considered for the experiment, where the function J is taken as

$J(x) = (j_1(x),\ j_2(x),\ \dots,\ j_n(x))^T$.

Problem 1 [18] Exponential Function.

$$
\begin{aligned}
&j_1(x) = e^{x_1} - 1, \\
&j_i(x) = e^{x_i} + x_i - 1,\ \text{for } i = 2, 3, \dots, n, \\
&\text{and } \Lambda = \mathbb{R}^n_+.
\end{aligned}
$$

Problem 2 [18] Modified Logarithmic Function.

$$
\begin{aligned}
&j_i(x) = \ln(x_i + 1) - \frac{x_i}{n},\ \text{for } i = 2, 3, \dots, n, \\
&\text{and } \Lambda = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i \le n, x_i > -1, i = 1, 2, \dots, n\}.
\end{aligned}
$$

Problem 3 [18] Strictly Convex Function I.

$$
\begin{aligned}
&j_i(x) = e^{x_i} - 1,\ \text{for } i = 1, 2, \dots, n, \\
&\text{and } \Lambda = \mathbb{R}^n_+.
\end{aligned}
$$

Problem 4

$$
\begin{aligned}
&j_i(x) = \frac{i}{n} e^{x_i} - 1,\ \text{for } i = 1, 2, \dots, n, \\
&\text{and } \Lambda = \mathbb{R}^n_+.
\end{aligned}
$$

Problem 5 [7] Tridiagonal Exponential Function.

$$
\begin{aligned}
j_1(x) &= x_1 - e^{\cos(h(x_1 + x_2))}, \\
j_i(x) &= x_i - e^{\cos(h(x_{i-1} + x_i + x_{i+1}))},\ \text{for } i = 2, \dots, n-1, \\
j_n(x) &= x_n - e^{\cos(h(x_{n-1} + x_n))}, \\
h &= \frac{1}{n+1} \text{ and } \Lambda = \mathbb{R}^n_+.
\end{aligned}
$$

Problem 6 [32] Nonsmooth Function.

$$
\begin{aligned}
&j_i(x) = x_i - \sin|x_i - 1|,\ \ i = 1, 2, 3, \dots, n. \\
&\text{and } \Lambda = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i \le n, x_i \ge -1, i = 1, 2, \dots, n\}.
\end{aligned}
$$

Problem 7 Pursuit-Evasion problem.

$$
\begin{aligned}
&j_i(x) = \sqrt{8}x_1 - 1,\ \ i = 1, 2, 3, \dots, n. \\
&\text{and } \Lambda = \mathbb{R}^n_+.
\end{aligned}
$$

The results of the experiments are tabulated in the following tables where ITER denotes the number of iterations, FVAL denotes the number of function evaluation, TIME denotes the CPU time and NORM denotes the norm of the function when an approximate solution is obtained. From the tables, it can be observed that all the three algorithms solved the seven test problems considered. However, our proposed SCD algorithm proved to be more efficient by solving most of the problems with less ITER, FVAL and TIME.

Table 1. Numerical Results of the **SCD**, **PDY** and **LLY** Algorithms on Problem 1 with given initial points and dimensions

		SCD				PDY				LLY			
DIMENSION	INITIAL POINT	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	x1	1	3	0.0023	0.00E+00	24	71	0.0552	7.80E-06	16	47	0.1044	6.27E-06
	x2	1	3	0.0026	0.00E+00	21	62	0.0250	8.99E-06	14	41	0.0233	9.50E-06
	x3	9	27	0.0065	5.95E-06	27	80	0.0211	7.48E-06	18	54	0.0327	7.62E-06
	x4	10	30	0.0121	2.20E-07	26	77	0.0269	8.98E-06	17	50	0.0301	8.58E-06
	x5	10	30	0.0083	2.13E-07	22	65	0.0326	5.66E-06	17	50	0.0348	8.19E-06
	x6	5	15	0.0048	5.31E-07	37	110	0.0416	8.03E-06	22	65	0.0286	6.98E-06
	x7	10	30	0.0111	2.20E-07	26	77	0.0267	8.98E-06	17	50	0.0214	8.58E-06
	x8	10	30	0.0097	1.99E-07	22	65	0.0216	5.67E-06	17	50	0.0187	8.25E-06
5000	x1	1	3	0.1281	0.00E+00	24	71	0.0649	7.57E-06	16	47	0.1032	5.24E-06
	x2	1	3	0.0051	0.00E+00	21	62	0.2491	9.56E-06	15	44	0.0707	8.28E-06
	x3	9	27	0.0214	5.95E-06	27	80	0.0522	7.48E-06	18	54	0.1663	7.62E-06
	x4	10	30	0.1192	4.67E-07	25	74	0.0595	7.95E-06	18	53	0.6757	7.35E-06
	x5	10	30	0.0356	4.63E-07	23	68	0.1996	6.33E-06	18	53	0.0589	7.26E-06
	x6	5	15	0.0151	5.36E-07	37	110	0.1605	8.02E-06	22	65	0.5717	7.00E-06
	x7	10	30	0.4784	4.67E-07	25	74	0.0927	7.95E-06	18	53	0.1644	7.35E-06
	x8	10	30	0.0508	4.57E-07	23	68	0.0572	6.34E-06	18	53	0.0556	7.27E-06
10000	x1	1	3	0.0882	0.00E+00	24	71	0.6136	8.67E-06	16	47	0.1069	5.63E-06
	x2	1	3	0.0119	0.00E+00	22	65	0.1816	6.24E-06	16	47	0.2273	4.62E-06
	x3	9	27	0.1883	5.95E-06	27	80	0.1676	7.48E-06	18	54	1.0679	7.62E-06
	x4	10	30	0.0738	6.56E-07	25	74	0.1600	7.11E-06	19	56	0.1169	4.09E-06
	x5	10	30	0.0415	6.53E-07	23	68	0.1979	8.96E-06	19	56	0.4385	4.06E-06
	x6	5	15	0.2782	5.37E-07	37	110	1.1591	8.02E-06	22	65	0.5424	7.00E-06
	x7	10	30	0.0558	6.56E-07	25	74	0.2842	7.11E-06	19	56	1.4820	4.09E-06
	x8	10	30	0.0568	6.49E-07	23	68	0.1104	8.96E-06	19	56	0.0980	4.06E-06
50000	x1	1	3	0.1613	0.00E+00	73	218	2.4883	9.05E-06	16	47	1.2844	9.30E-06
	x2	1	3	0.0400	0.00E+00	23	68	0.4548	5.67E-06	17	50	0.4646	4.07E-06
	x3	9	27	0.1168	5.95E-06	27	80	0.5052	7.48E-06	18	54	0.6553	7.62E-06
	x4	10	30	0.9183	1.46E-06	25	74	0.4628	7.52E-06	19	56	0.5190	9.09E-06
	x5	10	30	0.1708	1.46E-06	25	74	0.6266	5.01E-06	19	56	0.4309	9.08E-06
	x6	5	15	0.0923	5.37E-07	37	110	0.6462	8.02E-06	22	65	1.5992	7.00E-06
	x7	10	30	0.2276	1.46E-06	25	74	0.4413	7.52E-06	19	56	0.5693	9.09E-06
	x8	10	30	0.3395	1.45E-06	25	74	1.2421	5.01E-06	19	56	0.3643	9.08E-06
100000	x1	1	3	0.0782	0.00E+00	163	488	5.5216	9.42E-06	17	50	1.0319	4.97E-06
	x2	1	3	0.0638	0.00E+00	23	68	0.6252	7.60E-06	17	50	0.7126	5.76E-06
	x3	9	27	0.9063	5.95E-06	27	80	0.9798	7.48E-06	18	54	0.6714	7.62E-06
	x4	10	30	0.5563	2.06E-06	73	218	2.9124	9.73E-06	20	59	0.8634	5.08E-06
	x5	10	30	0.4564	2.06E-06	60	179	1.8583	8.93E-06	20	59	1.1861	5.07E-06
	x6	5	15	0.1634	5.37E-07	37	110	1.1625	8.02E-06	22	65	0.8905	7.00E-06
	x7	10	30	0.2648	2.06E-06	73	218	2.3206	9.73E-06	20	59	0.8795	5.08E-06
	x8	10	30	1.4084	2.06E-06	60	179	1.7355	8.93E-06	20	59	0.9313	5.07E-06

Table 2. Numerical Results of the **SCD**, **PDY** and **LLY** Algorithms on Problem 2 with given initial points and dimensions

		SCD				PDY				LLY			
DIMENSION	INITIAL POINT	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	x1	7	20	0.0887	3.54E-06	5	9	0.0165	3.60E-08	9	25	0.0173	9.20E-06
	x2	6	17	0.0108	9.88E-06	3	5	0.0068	5.17E-07	7	19	0.0114	2.66E-06
	x3	7	19	0.0106	7.53E-06	18	50	0.0253	5.39E-06	9	25	0.0101	4.43E-06
	x4	9	25	0.0161	4.48E-06	22	60	0.0337	7.42E-06	28	82	0.0440	3.94E-06
	x5	9	25	0.0175	4.48E-06	22	60	0.0206	7.42E-06	28	82	0.0445	3.94E-06
	x6	8	22	0.0142	5.70E-06	19	53	0.0319	7.68E-06	21	61	0.0281	4.81E-06
	x7	9	25	0.0363	4.48E-06	22	60	0.0583	7.42E-06	28	82	0.1102	3.94E-06
	x8	9	25	0.0156	4.49E-06	25	66	0.0886	6.32E-06	28	82	0.2167	3.75E-06
5000	x1	7	20	0.1270	8.48E-06	5	9	0.2202	6.26E-09	10	28	0.1341	3.18E-06
	x2	7	19	0.1724	9.33E-06	3	5	0.0160	1.75E-07	7	19	0.0248	5.70E-06
	x3	7	19	0.0422	7.72E-06	18	50	0.0524	5.37E-06	9	25	0.0287	3.98E-06
	x4	9	26	0.2498	2.08E-06	23	66	0.0857	5.26E-06	30	88	1.0625	3.86E-06
	x5	9	26	0.2010	2.08E-06	23	66	0.4698	5.26E-06	30	88	0.1174	3.86E-06
	x6	8	22	0.2322	6.45E-06	19	53	0.6032	7.43E-06	21	61	0.3431	6.36E-06
	x7	9	26	0.0440	2.08E-06	23	66	0.0918	5.26E-06	30	88	0.3690	3.86E-06
	x8	9	26	0.0336	2.08E-06	23	66	0.0886	5.26E-06	30	88	0.1348	3.87E-06
10000	x1	8	22	0.2472	4.84E-06	5	9	0.6819	3.62E-09	10	28	0.0891	4.48E-06
	x2	7	20	0.5338	2.65E-06	3	5	0.1175	1.21E-07	7	19	0.0432	8.03E-06
	x3	7	19	0.1134	7.74E-06	18	50	0.6410	5.37E-06	10	28	1.0279	4.40E-06
	x4	9	26	0.3269	2.95E-06	23	66	0.7092	7.43E-06	30	88	0.2219	5.48E-06
	x5	9	26	0.1460	2.95E-06	23	66	0.2032	7.43E-06	30	88	0.8150	5.48E-06
	x6	8	22	0.2239	6.55E-06	19	53	0.1378	7.40E-06	21	61	0.1314	6.51E-06
	x7	9	26	0.1494	2.95E-06	23	66	0.3822	7.43E-06	30	88	0.1683	5.48E-06
	x8	9	26	0.6483	2.96E-06	23	66	0.5253	7.43E-06	30	88	0.1853	5.49E-06
50000	x1	8	23	0.4542	2.18E-06	26	77	1.0290	7.75E-06	10	28	0.7978	9.97E-06
	x2	7	20	0.3154	5.97E-06	3	5	0.0983	6.32E-08	8	22	0.2194	2.86E-06
	x3	7	19	0.6218	7.76E-06	18	50	0.3373	5.36E-06	11	31	0.7054	8.67E-06
	x4	9	26	0.4564	6.64E-06	24	69	1.4849	8.30E-06	32	94	2.7029	4.59E-06
	x5	9	26	1.1128	6.64E-06	24	69	0.4811	8.30E-06	32	94	0.9953	4.59E-06
	x6	8	22	0.1500	6.63E-06	19	53	0.5591	7.37E-06	21	61	0.5683	6.63E-06
	x7	9	26	0.5885	6.64E-06	24	69	0.5904	8.30E-06	32	94	1.5171	4.59E-06
	x8	9	26	0.1896	6.64E-06	24	69	1.0501	8.30E-06	32	94	1.0207	4.59E-06
100000	x1	8	23	0.8410	3.08E-06	63	188	2.8714	7.77E-06	11	31	0.6148	2.26E-06
	x2	7	20	0.3791	8.45E-06	3	5	0.1158	5.40E-08	8	22	0.6978	4.04E-06
	x3	7	19	0.7152	7.76E-06	18	50	1.7273	5.36E-06	13	37	0.8367	3.15E-06
	x4	9	26	0.5317	9.39E-06	26	77	1.1706	5.19E-06	32	94	2.1078	6.49E-06
	x5	9	26	0.5580	9.39E-06	26	77	1.2016	5.19E-06	32	94	2.1205	6.49E-06
	x6	8	22	0.6292	6.64E-06	19	53	0.9325	7.36E-06	21	61	1.2519	6.65E-06
	x7	9	26	1.6351	9.39E-06	26	77	1.2891	5.19E-06	32	94	2.1439	6.49E-06
	x8	9	26	0.4745	9.39E-06	26	77	1.1348	5.19E-06	32	94	2.1427	6.49E-06

Table 3. Numerical Results of the **SCD**, **PDY** and **LLY** Algorithms on Problem 3 with given initial points and dimensions

		SCD				PDY				LLY			
DIMENSION	INITIAL POINT	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	x1	1	3	0.0162	0.00E+00	20	59	0.0298	7.37E-06	21	62	0.0354	6.57E-06
	x2	1	3	0.0018	0.00E+00	19	56	0.0182	5.45E-06	8	23	0.0138	2.78E-06
	x3	1	3	0.0062	2.22E-16	16	48	0.0131	5.62E-06	25	75	0.0253	8.99E-06
	x4	15	44	0.0069	8.85E-06	20	59	0.0233	8.76E-06	22	65	0.0222	5.38E-06
	x5	15	44	0.0070	8.85E-06	20	59	0.0414	8.76E-06	22	65	0.0406	5.38E-06
	x6	13	38	0.0636	7.75E-06	17	50	0.0235	6.76E-06	17	50	0.0187	9.63E-06
	x7	15	44	0.0124	8.85E-06	20	59	0.0125	8.76E-06	22	65	0.0197	5.38E-06
	x8	15	44	0.0125	8.94E-06	20	59	0.0120	8.77E-06	22	65	0.0217	5.42E-06
5000	x1	1	3	0.0222	0.00E+00	21	62	0.2091	8.24E-06	22	65	0.3812	7.28E-06
	x2	1	3	0.0041	0.00E+00	20	59	0.1648	6.10E-06	8	23	0.0397	6.21E-06
	x3	1	3	0.0049	2.22E-16	16	48	0.0373	5.62E-06	25	75	0.9425	8.99E-06
	x4	16	47	0.0427	6.99E-06	21	62	0.0509	9.80E-06	23	68	1.7854	5.99E-06
	x5	16	47	0.2445	6.99E-06	21	62	0.5293	9.80E-06	23	68	0.0538	5.99E-06
	x6	13	38	0.0480	7.75E-06	17	50	0.0411	6.76E-06	17	50	0.0412	9.63E-06
	x7	16	47	0.0308	6.99E-06	21	62	0.0425	9.80E-06	23	68	0.0838	5.99E-06
	x8	16	47	0.0378	7.01E-06	21	62	0.4519	9.80E-06	23	68	0.4262	6.00E-06
10000	x1	1	3	0.0104	0.00E+00	22	65	0.0911	5.83E-06	23	68	0.0913	5.11E-06
	x2	1	3	0.1023	0.00E+00	20	59	0.0625	8.62E-06	8	23	0.0547	8.79E-06
	x3	1	3	0.0114	2.22E-16	16	48	0.0557	5.62E-06	25	75	1.2883	8.99E-06
	x4	16	47	0.0475	9.89E-06	22	65	0.3407	6.93E-06	23	68	0.0952	8.47E-06
	x5	16	47	0.0495	9.89E-06	22	65	0.9084	6.93E-06	23	68	0.1117	8.47E-06
	x6	13	38	0.0592	7.75E-06	17	50	0.0576	6.76E-06	17	50	0.0766	9.63E-06
	x7	16	47	0.0475	9.89E-06	22	65	0.0776	6.93E-06	23	68	0.3569	8.47E-06
	x8	16	47	0.1758	9.90E-06	22	65	0.7398	6.93E-06	23	68	0.1712	8.48E-06
50000	x1	1	3	0.0287	0.00E+00	57	170	1.7362	8.87E-06	24	71	1.5008	5.67E-06
	x2	1	3	0.0194	0.00E+00	21	62	0.2404	9.64E-06	9	26	0.2185	3.14E-06
	x3	1	3	0.0236	2.22E-16	16	48	0.1844	5.62E-06	25	75	0.5480	8.99E-06
	x4	17	50	0.4714	7.79E-06	56	167	1.0036	7.97E-06	24	71	0.6973	9.40E-06
	x5	17	50	0.7310	7.79E-06	56	167	1.2997	7.97E-06	24	71	0.6250	9.40E-06
	x6	13	38	0.2250	7.75E-06	17	50	0.3479	6.76E-06	17	50	0.8318	9.63E-06
	x7	17	50	0.7120	7.79E-06	56	167	1.7737	7.97E-06	24	71	0.9931	9.40E-06
	x8	17	50	0.2993	7.79E-06	56	167	0.7052	7.97E-06	24	71	0.6573	9.40E-06
100000	x1	1	3	0.0709	0.00E+00	126	377	3.1485	9.15E-06	24	71	0.6936	8.01E-06
	x2	1	3	0.0773	0.00E+00	22	65	0.4317	6.82E-06	9	26	0.5340	4.45E-06
	x3	1	3	0.2186	2.22E-16	16	48	0.3266	5.62E-06	25	75	1.4667	8.99E-06
	x4	17	51	0.7640	6.06E-06	57	170	1.9869	8.46E-06	25	74	1.7901	6.59E-06
	x5	17	51	0.3707	6.06E-06	57	170	1.3907	8.46E-06	25	74	1.2399	6.59E-06
	x6	13	38	0.3836	7.75E-06	17	50	0.3276	6.76E-06	17	50	0.5267	9.63E-06
	x7	17	51	0.4190	6.06E-06	57	170	2.3809	8.46E-06	25	74	1.2321	6.59E-06
	x8	17	51	0.6001	6.06E-06	57	170	1.3800	8.46E-06	25	74	1.5502	6.59E-06

Table 4. Numerical Results of the **SCD**, **PDY** and **LLY** Algorithms on Problem 4 with given initial points and dimensions

		SCD				PDY				LLY			
DIMENSION	INITIAL POINT	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	x1	11	32	0.0329	4.63E-06	25	74	0.0168	5.22E-06	23	68	0.0270	9.92E-06
	x2	16	44	0.0171	9.38E-06	25	68	0.0149	7.57E-06	28	79	0.0285	8.26E-06
	x3	31	89	0.0455	9.48E-06	24	65	0.0402	5.51E-06	26	73	0.0334	3.80E-06
	x4	15	42	0.0148	6.35E-06	24	65	0.0214	8.15E-06	24	67	0.0297	3.74E-06
	x5	17	50	0.0185	7.34E-06	28	83	0.0436	5.81E-06	26	76	0.1502	5.06E-06
	x6	30	87	0.0350	7.98E-06	25	67	0.0415	6.04E-06	30	85	0.0413	5.71E-06
	x7	15	42	0.0115	6.35E-06	24	65	0.0289	8.15E-06	24	67	0.0251	3.74E-06
	x8	17	50	0.0155	6.93E-06	28	83	0.0216	5.78E-06	26	76	0.0445	5.04E-06
5000	x1	11	32	0.0300	4.34E-06	27	80	0.8318	5.74E-06	25	74	0.0834	9.78E-06
	x2	13	36	0.1954	8.58E-07	26	70	0.0767	5.44E-06	28	79	0.0832	7.83E-06
	x3	13	36	0.0910	2.86E-06	26	70	0.1027	8.59E-06	20	55	1.0185	2.06E-06
	x4	30	86	0.4084	9.99E-06	24	65	0.2713	8.94E-06	25	70	0.0703	8.60E-06
	x5	18	53	0.1808	5.69E-06	30	89	0.1591	6.71E-06	27	79	0.0681	7.40E-06
	x6	18	50	0.4918	7.40E-06	30	79	0.2369	6.40E-06	26	73	0.7855	8.49E-06
	x7	30	86	0.0633	9.99E-06	24	65	0.6722	8.94E-06	25	70	0.0644	8.60E-06
	x8	18	53	0.0559	5.64E-06	30	89	0.3491	6.70E-06	27	79	0.1182	7.39E-06
10000	x1	11	32	0.0422	5.65E-06	28	83	0.1617	5.41E-06	26	77	0.3622	8.92E-06
	x2	12	32	0.3794	7.21E-06	26	70	0.1132	5.55E-06	25	70	0.4465	9.01E-06
	x3	12	32	0.0459	6.50E-06	26	70	0.9694	5.32E-06	29	82	0.3014	8.43E-06
	x4	31	90	0.1584	7.88E-06	25	68	0.1102	5.39E-06	30	86	1.9858	5.65E-06
	x5	18	53	0.3939	8.01E-06	31	92	0.4441	6.42E-06	28	82	0.2502	5.95E-06
	x6	20	56	0.0879	9.91E-06	27	73	0.1085	6.00E-06	29	82	0.2371	3.01E-06
	x7	31	90	0.0993	7.88E-06	25	68	0.3265	5.39E-06	30	86	1.1008	5.65E-06
	x8	18	53	0.5774	7.98E-06	31	92	0.1272	6.42E-06	28	82	0.1615	5.95E-06
50000	x1	11	33	0.1342	2.36E-06	75	224	1.4491	9.99E-06	28	83	2.0269	9.26E-06
	x2	12	32	0.1422	6.04E-06	34	101	1.2540	5.92E-06	32	92	0.5892	5.34E-06
	x3	12	32	0.7745	5.20E-06	34	101	0.6702	5.01E-06	31	88	0.9940	4.35E-06
	x4	18	50	0.2091	9.23E-06	27	74	0.4584	6.13E-06	31	89	0.5563	6.21E-06
	x5	19	56	0.4123	6.27E-06	80	239	2.1233	8.32E-06	31	91	2.1652	5.42E-06
	x6	12	32	0.4123	7.47E-06	33	98	0.4356	7.79E-06	32	91	0.6577	6.18E-06
	x7	18	50	0.4731	9.23E-06	27	74	0.3987	6.13E-06	31	89	0.7467	6.21E-06
	x8	19	56	0.2843	6.27E-06	80	239	1.7214	8.32E-06	31	91	0.8834	5.42E-06
100000	x1	11	33	0.2290	3.31E-06	78	233	2.4460	8.43E-06	29	86	1.8326	8.61E-06
	x2	12	32	0.5847	7.74E-06	35	104	0.9774	5.62E-06	32	92	1.3910	6.64E-06
	x3	12	32	0.3136	6.73E-06	34	101	1.0017	9.54E-06	32	92	1.4917	7.45E-06
	x4	18	51	0.4862	5.77E-06	33	98	1.1617	8.46E-06	30	86	1.2599	6.09E-06
	x5	19	56	1.3369	8.88E-06	82	245	2.3141	9.35E-06	31	92	1.2728	6.75E-06
	x6	12	32	0.3084	7.97E-06	34	101	0.9639	7.42E-06	32	92	1.2935	7.42E-06
	x7	18	51	0.3937	5.77E-06	33	98	1.1212	8.46E-06	30	86	1.5763	6.09E-06
	x8	19	56	0.4793	8.87E-06	82	245	3.9044	9.35E-06	31	92	1.3877	6.75E-06

Table 5. Numerical Results of the **SCD**, **PDY** and **LLY** Algorithms on Problem 5 with given initial points and dimensions

		SCD				PDY				LLY			
DIMENSION	INITIAL POINT	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	x1	7	20	0.0890	5.73E-06	23	68	0.1312	6.47E-06	16	47	0.0781	6.49E-06
	x2	7	20	0.0116	8.72E-06	23	68	0.0344	9.86E-06	16	47	0.0283	6.84E-06
	x3	7	20	0.0125	9.05E-06	24	71	0.0322	5.11E-06	15	44	0.0300	6.34E-06
	x4	7	20	0.0123	7.46E-06	23	68	0.0261	8.42E-06	15	44	0.0232	8.75E-06
	x5	7	20	0.0126	7.46E-06	23	68	0.1559	8.42E-06	15	44	0.0274	8.75E-06
	x6	7	20	0.0088	9.03E-06	24	71	0.0258	5.10E-06	15	44	0.0249	6.98E-06
	x7	7	20	0.0107	7.46E-06	23	68	0.1476	8.42E-06	15	44	0.0412	8.75E-06
	x8	7	20	0.0119	7.45E-06	23	68	0.0293	8.42E-06	15	44	0.0228	8.74E-06
5000	x1	7	21	0.0271	2.55E-06	24	71	0.2577	7.24E-06	12	35	0.1706	2.77E-06
	x2	7	21	0.0616	3.88E-06	25	74	0.1040	5.52E-06	12	35	0.1578	4.56E-06
	x3	7	21	0.0304	4.03E-06	25	74	0.1218	5.73E-06	12	35	1.1935	3.25E-06
	x4	7	21	0.2711	3.32E-06	24	71	0.1051	9.43E-06	12	35	0.1123	5.18E-06
	x5	7	21	0.0268	3.32E-06	24	71	0.1291	9.43E-06	12	35	0.0652	5.18E-06
	x6	7	21	0.0999	4.03E-06	25	74	0.0876	5.72E-06	12	35	0.0821	1.75E-06
	x7	7	21	0.0388	3.32E-06	24	71	0.2387	9.43E-06	12	35	0.0664	5.18E-06
	x8	7	21	0.1139	3.32E-06	24	71	0.1726	9.43E-06	12	35	0.0552	5.17E-06
10000	x1	7	21	0.0726	3.60E-06	25	74	0.2084	5.12E-06	11	32	2.6108	9.22E-06
	x2	7	21	0.2388	5.49E-06	60	179	0.4705	8.35E-06	12	35	0.1240	8.05E-06
	x3	7	21	0.0838	5.70E-06	60	179	0.4566	8.67E-06	12	35	0.1646	2.72E-06
	x4	7	21	0.2351	4.69E-06	59	176	0.9589	9.51E-06	15	44	0.2307	2.15E-06
	x5	7	21	0.0911	4.69E-06	59	176	0.3671	9.51E-06	15	44	1.4382	2.15E-06
	x6	7	21	0.2526	5.70E-06	60	179	0.4100	8.67E-06	12	35	0.1004	3.33E-06
	x7	7	21	0.0543	4.69E-06	59	176	0.7330	9.51E-06	15	44	0.1264	2.15E-06
	x8	7	21	0.0482	4.69E-06	59	176	0.4571	9.51E-06	15	44	0.2378	2.15E-06
50000	x1	7	21	0.5848	8.06E-06	61	182	2.6156	9.19E-06	11	32	2.0915	4.97E-06
	x2	8	23	0.2971	4.91E-06	134	401	3.3776	9.92E-06	11	32	0.7214	7.74E-06
	x3	8	23	0.4958	5.10E-06	135	404	3.2328	9.01E-06	11	32	0.5964	7.43E-06
	x4	8	23	0.5933	4.20E-06	133	398	3.0908	9.69E-06	12	35	0.6424	2.81E-06
	x5	8	23	0.3562	4.20E-06	133	398	3.3884	9.69E-06	12	35	0.6049	2.81E-06
	x6	8	23	0.2888	5.10E-06	135	404	3.3171	9.01E-06	11	32	0.3994	6.89E-06
	x7	8	23	0.2346	4.20E-06	133	398	3.2587	9.69E-06	12	35	0.6985	2.81E-06
	x8	8	23	0.1696	4.20E-06	133	398	3.3766	9.69E-06	12	35	1.7276	2.81E-06
100000	x1	8	23	0.5196	4.56E-06	134	401	6.5738	9.21E-06	11	32	1.6355	2.69E-06
	x2	8	23	0.3381	6.95E-06	283	848	20.1453	9.68E-06	11	32	1.2017	4.13E-06
	x3	8	23	0.5179	7.21E-06	284	851	14.4649	9.42E-06	11	32	1.1021	4.20E-06
	x4	8	23	0.9699	5.93E-06	136	407	7.4457	9.18E-06	11	32	1.0706	6.59E-06
	x5	8	23	0.5388	5.93E-06	136	407	6.8313	9.18E-06	11	32	0.8234	6.59E-06
	x6	8	23	0.5458	7.21E-06	284	851	13.6980	9.42E-06	11	32	1.2348	4.13E-06
	x7	8	23	0.4051	5.93E-06	136	407	6.5345	9.18E-06	11	32	1.2016	6.59E-06
	x8	8	23	0.8464	5.93E-06	136	407	6.6888	9.18E-06	11	32	0.8019	6.59E-06

Table 6. Numerical Results of the **SCD**, **PDY** and **LLY** Algorithms on Problem 6 with given initial points and dimensions

		SCD				PDY				LLY			
DIMENSION	INITIAL POINT	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	x1	10	30	0.0430	6.68E-06	6	17	0.0103	6.75E-07	6	17	0.0164	3.52E-06
	x2	10	29	0.0109	4.57E-06	6	17	0.0087	3.28E-06	6	17	0.0259	6.62E-06
	x3	10	29	0.0144	4.79E-06	24	71	0.1287	5.35E-06	20	59	0.0222	9.01E-06
	x4	11	32	0.0121	3.96E-06	23	68	0.0292	9.21E-06	20	59	0.0293	4.39E-06
	x5	11	32	0.0139	3.96E-06	23	68	0.0303	9.21E-06	20	59	0.0206	4.39E-06
	x6	10	29	0.0132	4.87E-06	20	59	0.0714	9.10E-06	17	50	0.0216	6.74E-06
	x7	11	32	0.0080	3.96E-06	23	68	0.0376	9.21E-06	20	59	0.0279	4.39E-06
	x8	11	32	0.0122	3.97E-06	23	68	0.0274	9.23E-06	20	59	0.0179	4.40E-06
5000	x1	11	32	0.0895	4.87E-06	6	17	0.9856	1.51E-06	6	17	0.0242	7.86E-06
	x2	10	30	0.0597	6.69E-06	6	17	0.0333	7.33E-06	7	20	0.8547	8.34E-07
	x3	10	30	0.1727	7.00E-06	18	53	0.0545	9.70E-06	17	50	0.1656	4.70E-06
	x4	11	32	0.0399	8.86E-06	25	74	0.0773	5.83E-06	20	59	0.0785	9.82E-06
	x5	11	32	0.1033	8.86E-06	25	74	0.1119	5.83E-06	20	59	0.0682	9.82E-06
	x6	10	30	0.0278	7.03E-06	21	62	0.0946	7.52E-06	19	56	0.5985	6.90E-06
	x7	11	32	0.1430	8.86E-06	25	74	0.1078	5.83E-06	20	59	0.0861	9.82E-06
	x8	11	32	0.0289	8.86E-06	25	74	0.0929	5.84E-06	20	59	0.0803	9.83E-06
10000	x1	11	32	0.0476	6.89E-06	6	17	0.0389	2.13E-06	7	20	0.1593	6.27E-07
	x2	10	30	0.0469	9.46E-06	7	20	0.0532	6.62E-07	7	20	0.4635	1.18E-06
	x3	10	30	0.1110	9.91E-06	16	47	0.1259	2.57E-06	17	50	0.0965	4.92E-06
	x4	11	33	0.0534	8.20E-06	25	74	0.4258	8.25E-06	21	62	0.1299	6.03E-06
	x5	11	33	0.1411	8.20E-06	25	74	0.9868	8.25E-06	21	62	1.1701	6.03E-06
	x6	10	30	0.0433	9.92E-06	25	74	0.2755	7.73E-06	21	62	0.1347	7.80E-06
	x7	11	33	0.1191	8.20E-06	25	74	0.3708	8.25E-06	21	62	0.3222	6.03E-06
	x8	11	33	0.0481	8.20E-06	25	74	0.1590	8.25E-06	21	62	0.6405	6.03E-06
50000	x1	12	35	0.4852	3.28E-06	27	80	0.4618	7.69E-06	7	20	0.2612	1.40E-06
	x2	11	32	0.7793	6.89E-06	7	20	0.5416	1.48E-06	7	20	0.2092	2.64E-06
	x3	11	32	0.2076	7.22E-06	21	62	0.4876	6.85E-06	17	50	0.7491	6.82E-06
	x4	12	35	0.1999	5.97E-06	26	77	0.5229	9.81E-06	22	65	0.6362	5.85E-06
	x5	12	35	0.3531	5.97E-06	26	77	0.6792	9.81E-06	22	65	2.1945	5.85E-06
	x6	11	32	0.1696	7.22E-06	20	59	1.3486	8.01E-06	18	53	1.5017	7.51E-06
	x7	12	35	0.5930	5.97E-06	26	77	1.2159	9.81E-06	22	65	1.0642	5.85E-06
	x8	12	35	0.4212	5.97E-06	26	77	0.4800	9.81E-06	22	65	0.4915	5.85E-06
100000	x1	12	35	0.3894	4.64E-06	28	83	0.9476	5.79E-06	7	20	1.4489	1.98E-06
	x2	11	32	1.1153	9.75E-06	28	83	1.5076	5.43E-06	7	20	0.4727	3.73E-06
	x3	11	33	0.3290	6.68E-06	28	83	1.0226	7.06E-06	17	50	1.2814	8.73E-06
	x4	12	35	0.9861	8.45E-06	27	80	1.0002	6.81E-06	22	65	1.2254	8.27E-06
	x5	12	35	0.2797	8.45E-06	27	80	0.7974	6.81E-06	22	65	1.2500	8.27E-06
	x6	11	33	1.0946	6.68E-06	28	83	1.8733	7.06E-06	18	53	0.8855	7.57E-06
	x7	12	35	0.4250	8.45E-06	27	80	0.9973	6.81E-06	22	65	1.3610	8.27E-06
	x8	12	35	0.6744	8.45E-06	27	80	1.0637	6.81E-06	22	65	1.3076	8.27E-06

Table 7. Numerical Results of the **SCD**, **PDY** and **LLY** Algorithms on Problem 7 with given initial points and dimensions

		SCD				PDY				LLY			
DIMENSION	INITIAL POINT	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	x1	7	21	0.0285	5.66E-06	13	38	0.0125	6.75E-06	9	26	0.0098	4.32E-06
	x2	7	20	0.0051	8.66E-06	12	35	0.0072	9.04E-06	9	26	0.0093	1.69E-06
	x3	7	21	0.0139	3.09E-06	13	38	0.0152	3.69E-06	9	26	0.0185	2.36E-06
	x4	7	21	0.0073	2.83E-06	13	38	0.0096	3.38E-06	9	26	0.0088	2.16E-06
	x5	7	21	0.0075	2.83E-06	13	38	0.0119	3.38E-06	9	26	0.0103	2.16E-06
	x6	7	21	0.0056	3.05E-06	13	38	0.0113	3.64E-06	9	26	0.0090	2.33E-06
	x7	7	21	0.0053	2.83E-06	13	38	0.0128	3.38E-06	9	26	0.0062	2.16E-06
	x8	7	21	0.0075	2.84E-06	13	38	0.0106	3.38E-06	9	26	0.0070	2.16E-06
5000	x1	8	23	0.2014	4.92E-06	14	41	0.0237	4.42E-06	9	26	0.0186	9.66E-06
	x2	7	21	0.0851	4.96E-06	13	38	0.1749	5.92E-06	9	26	0.0445	3.79E-06
	x3	7	21	0.0335	6.92E-06	13	38	0.0585	8.25E-06	9	26	0.0193	5.28E-06
	x4	7	21	0.0290	6.34E-06	13	38	0.0320	7.56E-06	9	26	0.0971	4.84E-06
	x5	7	21	0.0105	6.34E-06	13	38	0.0343	7.56E-06	9	26	0.0494	4.84E-06
	x6	7	21	0.0918	6.89E-06	13	38	0.0269	8.22E-06	9	26	0.4760	5.26E-06
	x7	7	21	0.0347	6.34E-06	13	38	0.1156	7.56E-06	9	26	0.0774	4.84E-06
	x8	7	21	0.0706	6.34E-06	13	38	0.0825	7.56E-06	9	26	0.0292	4.84E-06
10000	x1	8	23	0.0256	6.96E-06	14	41	0.3006	6.25E-06	10	29	0.0773	1.98E-06
	x2	7	21	0.0321	7.02E-06	13	38	0.0394	8.37E-06	9	26	0.0928	5.36E-06
	x3	7	21	0.0219	9.79E-06	14	41	0.0437	3.42E-06	9	26	0.6629	7.47E-06
	x4	7	21	0.2105	8.96E-06	14	41	0.0442	3.13E-06	9	26	0.1134	6.84E-06
	x5	7	21	0.0215	8.96E-06	14	41	0.1545	3.13E-06	9	26	0.0547	6.84E-06
	x6	7	21	0.0213	9.77E-06	14	41	0.0483	3.41E-06	9	26	0.2624	7.46E-06
	x7	7	21	0.0936	8.96E-06	14	41	0.0731	3.13E-06	9	26	0.9003	6.84E-06
	x8	7	21	0.2350	8.96E-06	14	41	0.1291	3.13E-06	9	26	0.3378	6.84E-06
50000	x1	8	24	0.1129	3.99E-06	41	122	1.0051	6.97E-06	10	29	0.5945	4.42E-06
	x2	8	23	0.3362	6.11E-06	14	41	0.1410	5.48E-06	10	29	0.2369	1.73E-06
	x3	8	23	0.0782	8.51E-06	14	41	0.1532	7.65E-06	10	29	1.5064	2.42E-06
	x4	8	23	0.0707	7.80E-06	14	41	0.3986	7.00E-06	10	29	0.2640	2.21E-06
	x5	8	23	0.0840	7.80E-06	14	41	0.5471	7.00E-06	10	29	0.4523	2.21E-06
	x6	8	23	0.2947	8.51E-06	14	41	0.2980	7.64E-06	10	29	1.2022	2.42E-06
	x7	8	23	0.0799	7.80E-06	14	41	0.5583	7.00E-06	10	29	0.2554	2.21E-06
	x8	8	23	0.3384	7.80E-06	14	41	0.2536	7.00E-06	10	29	0.3382	2.21E-06
100000	x1	8	24	0.2210	5.64E-06	41	122	1.3457	9.86E-06	10	29	0.2277	6.25E-06
	x2	8	23	0.3563	8.64E-06	14	41	0.2428	7.75E-06	10	29	0.4301	2.45E-06
	x3	8	24	0.1647	3.09E-06	15	44	0.9632	3.17E-06	10	29	0.9843	3.42E-06
	x4	8	24	0.2206	2.82E-06	14	41	0.3214	9.90E-06	10	29	0.6343	3.13E-06
	x5	8	24	0.1516	2.82E-06	14	41	0.5326	9.90E-06	10	29	0.4172	3.13E-06
	x6	8	24	0.1568	3.08E-06	15	44	0.3448	3.17E-06	10	29	0.3376	3.42E-06
	x7	8	24	0.4245	2.82E-06	14	41	0.3345	9.90E-06	10	29	1.2244	3.13E-06
	x8	8	24	0.1547	2.82E-06	14	41	0.2437	9.90E-06	10	29	0.3299	3.13E-06

Moreover, using the Dolan and Morè performance profile [12], we plot the graphs of the three algorithms in order to visualize their performance. The performance is shown in Figures 1, 2 and 3.

It can be observed from Figure 1 and 2 that the SCD algorithm outperformed the PDY and LLY algorithms by solving around 90% of the problems with less number of iterations and function evaluations. Furthermore, with regards to time, Figure 3 shows that the SCD algorithm solved around 70% of the problems with less time. Therefore, we can conclude that the proposed algorithm is more efficient than the PDY and LLY algorithms in terms of number of iterations, function evaluations as well as CPU time.

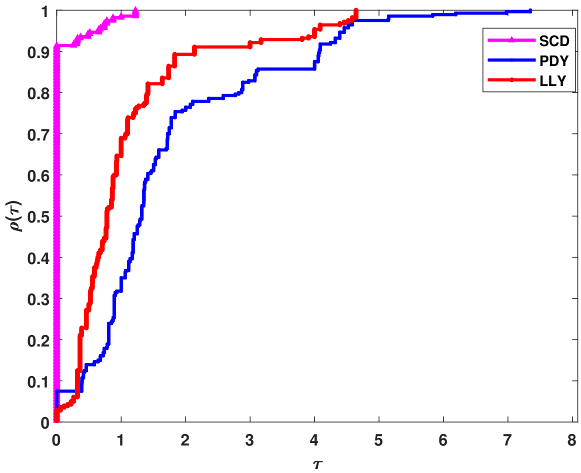

Fig. 1. Performance profile on number of iterations

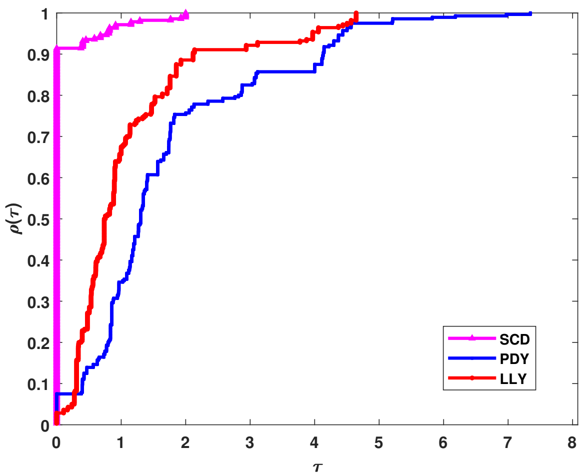

Fig. 2. Performance profile on function evaluations

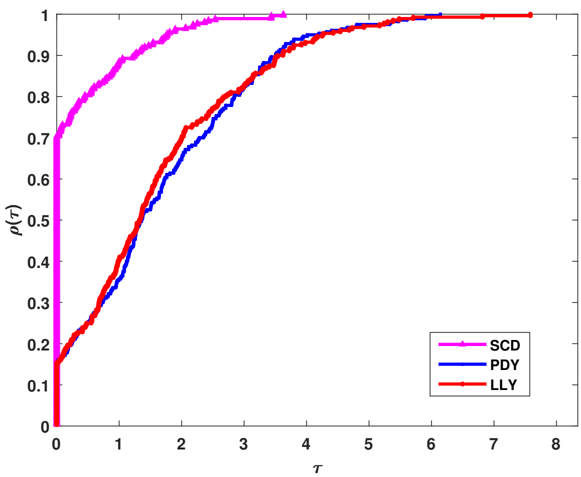

Fig. 3. Performance profile on CPU time

4. Application in Compressive Sensing

The problem of sparse signal reconstruction involves solving minimization of an objective function:

$$\min_x \frac{1}{2}\|Mx - y\|_2^2 + \rho\|x\|_1, \tag{4.1}$$

where $x \in R^n$, $q \in R^m$, $M \in R^{m\times n}$ $(m << n)$ is a linear operator, $\rho \geq 0$, $\|x\|_2$ is the Euclidean norm of x and $\|x\|_1 = \sum_{i=1}^n |x_i|$ is the ℓ_1-norm of x.

This problem is of interest to many researchers in signal processing. Some of the popular methods for solving (4.1) can be found in [13, 16, 5, 14, 27]. First, a reformulation of problem (4.1) into a quadratic problem was given by Figueiredo et al. [14]. They expressed $x \in \mathbb{R}^n$ in two parts

$$x = b - y, \qquad b \geq 0, \quad y \geq 0,$$

where $b_i = (x_i)_+$, $y_i = (-x_i)_+$ for all $i = 1, 2, ..., n$, and $(.)_+ = \max\{0, .\}$. Also, we have $\|x\|_1 = e_n^T b + e_n^T y$, where $e_n = (1, 1, ..., 1)^T \in R^n$. From this reformulation, equation (4.1) can be written as

$$\min_{b,y} \frac{1}{2}\|q - M(b - y)\|_2^2 + \rho e_n^T b + \rho e_n^T y, \qquad b \geq 0, \quad y \geq 0, \tag{4.2}$$

from [14], equation (4.2) can be written as

$$\min_z \frac{1}{2} z^T E z + c^T z, \qquad \text{such that} \quad z \geq 0, \tag{4.3}$$

where $z = \begin{pmatrix} b \\ y \end{pmatrix}$, $\quad c = \omega e_{2n} + \begin{pmatrix} -a \\ a \end{pmatrix}$, $\quad a = M^T q$, $\quad E = \begin{pmatrix} M^T M & -M^T M \\ -M^T M & M^T M \end{pmatrix}$.

It can be observed that E is a positive semi-definite showing that Problem (4.3) is quadratic programming problem.

Moreover, a translation of (4.3) into a linear variable problem, equivalently, a linear complementary problem was given by Xiao et al [30], and the variable z solves the linear complementary problem provided that it solves the nonlinear equation:

$$M(z) = \min\{z, Ez + c\} = 0, \tag{4.4}$$

where M is a vector-valued function. The mapping $M(z)$ is continuous and monotone as shown in [29, 23]. This implies problem (4.1) is equivalent to problem (1.1). Hence, the proposed SCD algorithm for solving (1.1) can be applicable to solve (4.1).

In this section, our proposed SCD algorithm is applied to restore a signal of length n from k observations. The performance of the SCD is compared with two existing methods, specifically, the CGD and PCG algorithms proposed in [21] and [30] respectively. We choose the parameters in the SCD algorithm as follows: $\beta = 10^{-5}$, $\alpha = 0.03$, $\sigma = 0.1$, $\kappa = 1$ and $c = 1$. However, in the CGD and PCG algorithms, the parameters are maintained as reported in the respective papers [21] and [30]. We run each of the three algorithms with same initial point and continuation technique on parameter μ. The convergence behviour of the algorithms is observed to obtain a solution with similar accuracy. For initialization, we used $x_0 = M^T y$ and the iterations were stopped when the inequality

$$\left| \frac{f(x_k) - f(x_{k-1})}{f(x_{k-1})} \right| < 10^{-5}.$$

is satisfied. To understand the quality of the restoration, mean squared error (MSE) is used as a metric. The MSE is given as:

$$MSE = \frac{1}{n} \|\widehat{x} - x\|^2,$$

In the experiment, M is a Gaussian matrix with $n = 2^{12}$, $k = 2^{10}$, the original signal contains 2^7 nonzero elements. We recover the original signal $\widehat{x}$ from y by $y = 2^{10}$ observations, and set $f(x) = \frac{1}{2}\|Mx - y\|_2^2 + \rho\|x\|_1$. Where $y = M\widehat{x} + \omega$, and ω is the Gaussian noise distributed as $N(0, 10^{-4})$.

Figures 4 and 5 depicts the perfomance of all the three methods. All the three successfully restored the signal. However, it can be observed that our SCD method has some advantages over the CGD and PCG methods. These advantages include lesser MSE, number of iterations and CPU time. In addition, four graphs are plotted to demonstrate the convergence behaviour of the algorithms. In this case also, our proposed SCD method is reported to have faster convergence rate in comparison with the other two. Thus, our proposed SCD method can efficiently restore a sparse signal with less error, in fewer iterations as well as CPU time.

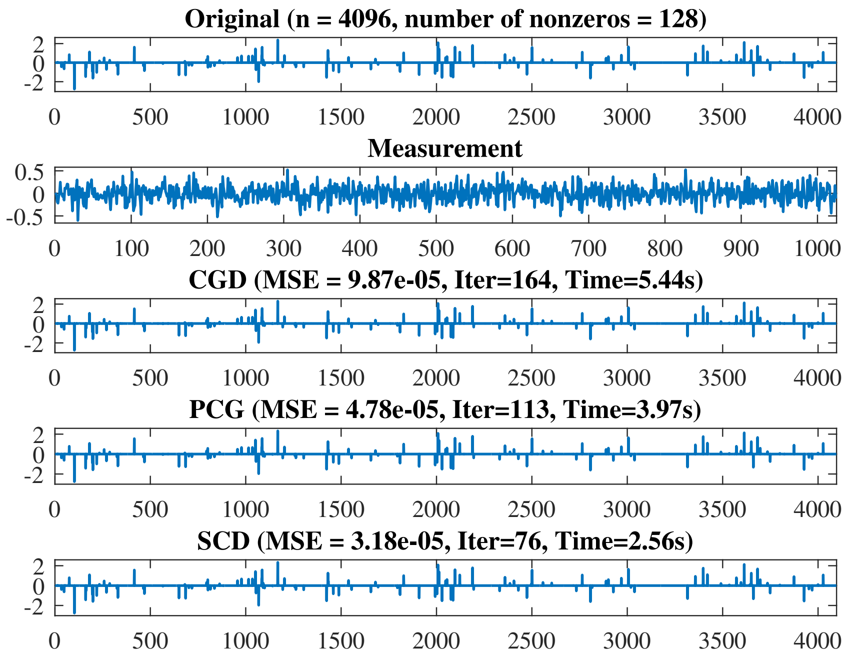

Fig. 4. From top to bottom: the original signal, the measurement, and the recovery signals by CGD, PCG and SCD methods.

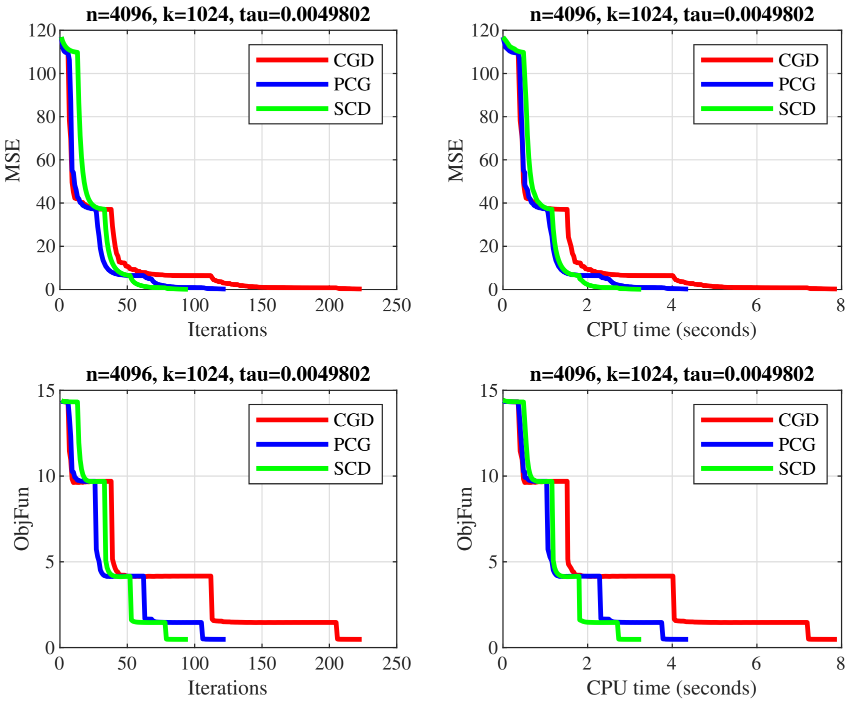

Fig. 5. Comparison of CGD, PCG and SCD methods based on MSE, number of iterations, objective function and CPU time.

5. Conclusion

In this paper, a derivative-free algorithm for solving systems of nonlinear monotone equations is proposed. The algorithm combined the well-known conjugate gradient method with a projection method. The global convergence of the method is established by assuming that the operator under study is Lipschitz continuous and monotone. Numerical experiments are performed to show the efficiency of the algorithm in comparison with some existing works, specifically, the algorithms proposed in [20] and [33]. The numerical results reported in this work have proved that the proposed algorithm is more efficient than the ones proposed in [20] and [33]. Furthermore, the proposed algorithm is applied in signal recovery problems, and the recovery result is compared with the PCG and CGD methods proposed in [21] and [30] respectively. The proposed algorithm is shown to recover the distorted signal with less MSE, number of iterations as well as CPU time, proving its effectiveness over the compared algorithms.

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Generalized Projection Algorithm for Convex Feasibility Problems on Hadamard Manifolds

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ABSTRACT

We present a two-step cyclic algorithm for solving convex feasibility problems on Hadamard manifolds in this study. On Hadamard manifolds, the convergent result and linear convergent results are proven. In addition, to support the main results, a numerical example on the Poincaré plane is provided.

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1. Introduction

Let H be a Hilbert space and $C_1, \dots, C_m$ are closed convex subsets of H with nonempty intersection $\bigcap_{i=1}^{m} C_i$. Finding a point at the intersection of convex sets is a challenge in mathematics and physical sciences. The problem is known as a convex feasibility problem, and it is defined as follows:

$$\text{Find a point } x \in \bigcap_{i=1}^{m} C_i.$$

A point x solving this problem is said to be a *feasibility point*. The *projection algorithm*, in which each iterative step is to project onto an individual set corresponding to a control sequence (see, for example, [7] for the definition of the control sequence), is one of the most widely studied methods for determining such feasibility points. For more information, see [4, 3, 13, 12, 28, 22, 2] and the references therein. Convex inequalities [17, 18], convex minimization problems [32, 26, 31], medical imaging [8] and computerized tomography [24, 1] are some of the applications of the projection method.

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Many nonlinear problems, such as fixed point theory, convex analysis, variational inequalities, equilibrium problems, and optimization problems, have been extended from linear spaces to the setting of manifolds in the last decade because the problems cannot be posted in linear space and necessarily require a Riemannian manifold structure, see for examples [30, 19, 23, 15, 5, 11, 27, 20, 10] and the reference therein.

Returning to the convex feasibility problems, Bauschke et al. [4] proposed the following general procedure with an initial point $x_0 \in H$ in Hilbert spaces:

$$x_{n+1} := (1-\alpha_n)x_n + \alpha_n \sum_{i=1}^{m} \mu_i^{(n)} T_i^{(n)}(x_n), \quad \forall n \in \mathbb{N}, \tag{1.1}$$

where each $\alpha_n \in [0,2]$ is a relaxation parameter, $\{\mu_i^{(n)} : 1, \dots, m\} \subseteq [0,1]$ is weight satisfying $\sum_{i=1}^{m} \mu_i^{(n)} = 1$, and each $T_i^{(n)} : H \to H$ is a firmly nonexpansive satisfying $\operatorname{Fix} T_i^{(n)} \supseteq C_i$. Some works [7, 16, 21] discuss the case where the weights $\left\{\mu_i^{(n)}\right\}$ satisfy the condition that

$$\mu_i^{(n)} = \delta_{i_n,i} := \begin{cases} 1, & \text{if } i = i_n, \\ 0, & \text{otherwise,} \end{cases}$$

where $\{i_n\}_{n=0}^{\infty}$ is a so-called *control sequence*, and each $T_i^{(n)}$ is the projection onto a hyperplane separating C_{i_n} from x_n. Censor [9] proposed the cyclic subgradient projection algorithm for the case where each convex is given as a sublevel set of a convex function, i.e., the algorithm uses weights $\{\mu_i^{(n)}\}$ satisfying the following condition

$$\mu_i^{(n)} := \begin{cases} 1, & \text{if } i = n \pmod{m} + 1, \\ 0, & \text{otherwise,} \end{cases} \tag{1.2}$$

and each $T_i^{(n)}$ is the projection operator of the corresponding subgradient. In the Hadamard manifolds, Wang et al. [30] extended the cyclic method (1.1) using the weights $\{\mu_i^{(n)}\}$ satisfying the condition (1.2). The following is the Riemannian version of the cyclic algorithm:

Algorithm 1.1. Let M be an Hadamard manifold and $x_0 \in M$ be an initial point. Define x_{n+1} by

$$x_{n+1} := \exp_{x_n} \alpha_n \exp_{x_n}^{-1} T_{i_n}^{(n)}(x_n), \quad \forall n \in \mathbb{N}, \tag{1.3}$$

where $T_{i_n}^{(n)} : M \to M$ be a family of firmly nonexpansive mapping, $\{\alpha_n\} \subseteq (0,2)$ is a relaxation parameter sequence, and set $i_n = n \pmod{m} + 1$. The authors [30] also established that Algorithm 1.1 is convergent and linearly convergent in Hadamard manifolds.

The goal of this paper is to extend the cyclic algorithm (1.3) to two-step projection cyclic algorithms on Hadamard manifolds, as motivated and inspired by the previous efforts. Furthermore, we show that Algorithm 3.1 is convergent under certain conditions. We show that this approach is linearly convergent when the algorithm is linearly focusing and the family of convex sets is linearly regular. On the Poincar'e plane, a numerical example is shown.

The rest of this paper is organized in the following: Section 2, we give some basic concept and fundamental results of Riemannian geometry. For solving convex feasibility problems, the two-step cyclic algorithm is described in Section 3. Furthermore, we show that any sequence generated by the proposed method converges to feasibility points. The linear convergence of the two-step projection cyclic projection algorithm is given in Section 4. Finally, Section 5 gives a numerical example of our method for approximating convex feasibility problem solutions on the Poincaré plane.

2. Preliminaries

In this section, we recall some fundamental definitions, properties, useful results, and notations of Riemannian geometry. For more information, readers can consult several textbooks [25, 14, 29].

Let M be a connected finite-dimensional manifold. For $p \in M$, we denote T_pM the *tangent space* of M at p which is a vector space of the same dimension as M, and by $TM = \bigcup_{p\in M} T_pM$ the *tangent bundle* of M. We always suppose that M can be endowed with a Riemannian metric $\langle \cdot, \cdot \rangle_p$, with corresponding norm denoted by $\| \cdot \|_p$, to become a *Riemannian manifold*. The angle $\angle_p(u, v)$ between $u, v \in T_pM$ $(u, v \neq \mathbf{0})$ is set by $\cos \angle_p(u, v) = \dfrac{\langle u, v\rangle_p}{\|u\|_p\|v\|_p}$. If there is no confusion, we denote $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_p$, $\| \cdot \| := \| \cdot \|_p$ and $\angle(u, v) := \angle_p(u, v)$. Let $\gamma : [a, b] \to M$ be a piecewise smooth curve joining $\gamma(a) = p$ to $\gamma(b) = q$, we define the length of the curve γ by using the metric as

$$L(\gamma) = \int_a^b \|\gamma^{'}(t)\| dt,$$

minimizing the length function over the set of all such curves, we obtain a Riemannian distance $d(p, q)$ which induces the original topology on M.

Let ∇ be a Levi-Civita connection associated to $(M, \langle \cdot, \cdot \rangle)$. Given γ a smooth curve, a smooth vector field X along γ is said to be *parallel* if $\nabla_{\gamma'} X = \mathbf{0}$, where $\mathbf{0}$ is the zero section of TM. If $\gamma^{'}$ itself is parallel, we say that γ is a *geodesic*, and in this case $\|\gamma^{'}\|$ is a constant. When $\|\gamma^{'}\| = 1$, then γ is said to be *normalized*. A geodesic joining p to q in M is said to be a *minimal geodesic* if its length equals to $d(p, q)$.

A Riemannian manifold is complete if for any $p \in M$ all geodesic emanating from p are defined for all $t \in \mathbb{R}$. From the Hopf-Rinow theorem we know that if M is complete then any pair of points in M can be joined by a minimal geodesic. Moreover, (M, d) is a complete metric space and every bounded closed subset is compact.

Let M be a complete Riemannian manifold and $p \in M$. The exponential map $\exp_p : T_pM \to M$ is defined as $\exp_p v = \gamma_v(1, x)$, where $\gamma(\cdot) = \gamma_v(\cdot, x)$ is the geodesic starting at p with velocity v (i.e., $\gamma_v(0, p) = p$ and $\gamma_v^{'}(0, p) = v$). Then, for any value of t, we have $\exp_p tv = \gamma_v(t, p)$ and $\exp_p \mathbf{0} = \gamma_v(0, p) = p$. Note that the exponential $\exp_p$ is differentiable on T_pM for all $p \in M$. It well known that the derivative $D\exp_p(\mathbf{0})$ of $\exp_p(\mathbf{0})$ is equal to the identity vector of T_pM. Therefore, by the inverse mapping theorem, there exists an inverse exponential map $\exp^{-1} : M \to T_pM$. Moreover, for any $p, q \in M$, we have $d(p, q) = \| \exp_p^{-1} q\|$.

A complete simply connected Riemannian manifold of non-positive sectional curvature is said to be an *Hadamard manifold.* Throughout the remainder of the paper, we always assume that M is a finite-dimensional Hadamard manifold. The following proposition is well-known and will be useful.

Proposition 2.1. [25] *Let $p \in M$. The $\exp_p : T_pM \to M$ is a diffeomorphism, and for any* *two points $p, q \in M$ there exists a unique normalized geodesic joining p to q, which is can be expressed by the formula*

$$\gamma(t) = \exp_p t \exp_p^{-1} q, \quad \forall t \in [0, 1].$$

This proposition yields that M is diffeomorphic to the Euclidean space $\mathbb{R}^n$. Then, M has same topology and differential structure as $\mathbb{R}^n$. Moreover, Hadamard manifolds and Euclidean spaces have some similar geometrical properties. One of the most important proprieties is illustrated in the following propositions.

A geodesic triangle $\triangle(p_1, p_2, p_3)$ of a Riemannian manifold M is a set consisting of three points p_1, p_2 and p_3, and three minimal geodesics γ_i joining p_i to p_{i+1} where $i = 1, 2, 3$ (mod3).

Proposition 2.2. [25] *Let $\triangle(p_1, p_2, p_3)$ be a geodesic triangle in Hadamard manifolds M. For each $i = 1, 2, 3$ (mod3), given $\gamma_i : [0, l_i] \to M$ the geodesic joining p_i to p_{i+1} and set $l_i := L(\gamma_i)$, $\alpha_i : \angle(\gamma_i'(0), -\gamma_{i-1}'(l_{i-1}))$. Then*

$$\alpha_1 + \alpha_2 + \alpha_3 \leq \pi; \tag{2.1}$$

$$l_i^2 + l_{i+1}^2 - 2 l_i l_{i+1} \cos \alpha_{i+1} \leq l_{i-1}^2. \tag{2.2}$$

In the terms of the distance and the exponential map, the inequality (2.2) can be rewritten as

$$d^2(p_i, p_{i+1}) + d^2(p_{i+1}, p_{i+2}) - 2\langle \exp_{p_{i+1}}^{-1} p_i, \exp_{p_{i+1}}^{-1} p_{i+2} \rangle \leq d^2(p_{i-1}, p_i), \tag{2.3}$$

where $\langle \exp_{p_{i+1}}^{-1} p_i, \exp_{p_{i+1}}^{-1} p_{i+2} \rangle = d(p_i, p_{i+1}) d(p_{i+1}, p_{i+2}) \cos \alpha_{i+1}$.

The following relation between geodesic triangles in Riemannian manifolds and triangles in $\mathbb{R}^2$ can be referred to [6].

Lemma 2.3. [6] *Let $\triangle(p_1, p_2, p_3)$ be a geodesic triangle in M. Then there exists a triangle $\triangle(\overline{p_1}, \overline{p_2}, \overline{p_3})$ for $\triangle(p_1, p_2, p_3)$ such that $d(p_i, p_{i+1}) = \|\overline{p_i} - \overline{p_{i+1}}\|$, indices taken modulo 3; it is unique up to an isometry of $\mathbb{R}^2$.*

The triangle $\triangle(\overline{p_1}, \overline{p_2}, \overline{p_3})$ in Lemma 2.3 is said to be a *comparison triangle* for $\triangle(p_1, p_2, p_3)$. The geodesic side from x to y will be denoted $[x, y]$. A point $\overline{x} \in [\overline{p_1}, \overline{p_2}]$ is said to be a *comparison point* for $x \in [p_1, p_2]$ if $\|\overline{x} - \overline{p_1}\| = d(x, p_1)$. The interior angle of $\triangle(\overline{p_1}, \overline{p_2}, \overline{p_3})$ at $\overline{p_1}$ is said to be the *comparison angle* between $\overline{p_2}$ and $\overline{p_3}$ at $\overline{p_1}$ and is denoted $\angle_{\overline{p_1}}(\overline{p_2}, \overline{p_3})$. With all notation as in the statement of Proposition 2.2, according to the law of cosine, (2.2) is valid if and only if

$$\langle \overline{p_2} - \overline{p_1}, \overline{p_3} - \overline{p_1} \rangle_{\mathbb{R}^2} \leq \langle \exp_{p_1}^{-1} p_2, \exp_{p_1}^{-1} p_3 \rangle \tag{2.4}$$

or,

$$\alpha_1 \leq \angle_{\overline{p_1}}(\overline{p_2}, \overline{p_3})$$

or, equivalent, $\triangle(p_1, p_2, p_3)$ satisfies the CAT(0) inequality and that is, given a comparison triangle $\overline{\triangle} \subset \mathbb{R}^2$ for $\triangle(p_1, p_2, p_3)$ for all $x, y \in \triangle$,

$$d(x, y) \leq \|\overline{x} - \overline{y}\|, \tag{2.5}$$

where $\overline{x}, \overline{y} \in \overline{\triangle}$ are the respective comparison points of x, y.

Definition 2.4. A subset Q is said to be *geodesic convex* if for any two points p and q in Q, the geodesic joining p to q is contained in Q, that is, if $\gamma : [a, b] \to M$ is a geodesic such that $p = \gamma(a)$ and $q = \gamma(b)$, then $\gamma((1 - t)a + tb) \in Q$ for all $t \in [0, 1]$.

Definition 2.5. A real function f defined on M is said to be *geodesic convex* if for any geodesic γ of M, the composition function $f \circ \gamma : [a, b] \to \mathbb{R}$ is convex, that is,

$$(f \circ \gamma)(ta + (1 - t)b) \leq t(f \circ \gamma)(a) + (1 - t)(f \circ \gamma)(b)$$

where $a, b \in \mathbb{R}$, and $t \in [0, 1]$.

Proposition 2.6. [25] *Let $d : M \times M \to \mathbb{R}$ be the distance function. Then d is a convex function with respect to the Riemannian metric, that is, for any pair of geodesics $\gamma_1 : [0, 1] \to M$ and $\gamma_2 : [0, 1] \to M$ the following inequality holds for all $t \in [0, 1]$*

$$d(\gamma_1(t), \gamma_2(t)) \leq (1 - t)d(\gamma_1(0)\gamma_2(0)) + td(\gamma_1(t), \gamma_2(t)).$$

In particular, for each $p \in M$, the function $d(\cdot, p) : M \to \mathbb{R}$ is a geodesic convex function.

A nonempty, closed geodesic convex set in M shall be denoted by Q from here on. Let $T : Q \to M$ be a mapping. We say that T is *nonexpansive* if for any two points $x, y \in Q$ such that

$$d(T(x), T(y)) \leq d(x, y).$$

Let $F(T)$ denote the set of all fixed points of T, i.e.,

$$F(T) := \{x \in Q : T(x) = x\}.$$

The definition of firmly nonexpansive mappings on Hadamard manifolds was introduce by Li et.al. [23].

Definition 2.7. [23] Let $T : Q \to M$ be a mapping. Then T is said to be *firmly nonexpansive* if for any $x, y \in Q$, the function $\sigma : [0, 1] \to [0, +\infty]$ defined by

$$\sigma(t) : d(\exp_x t \exp_x^{-1} T(x), \exp_x t \exp_y^{-1} T(y)), \quad \forall t \in [0, 1],$$

is nonincreasing.

By definition, it easy to see that any firmly nonexpansive mapping T is nonexpansive.

Next, we follow the definition of a *distance function* $d(\cdot, Q) : M \to \mathbb{R}$ and a *projection operator* $P_Q(\cdot) : M \to Q$, which are defined for every $x \in M$ by

$$d(x, Q) := \inf_{y \in Q} d(x, y)$$

and

$$P_Q(x) := \{z : d(x, z) \leq d(x, y), \quad \forall y \in Q\},$$

respectively. The projection operator P_Q is firmly nonexpansive as described in the following proposition.

Proposition 2.8. [23] *Let $Q \subseteq M$ be a nonempty, closed and convex set. Then the following assertions holds:*

(i) P_Q *is single valued and firmly nonexpansive;*

(ii) *For every* $x \in M$, $z = P_Q(x)$ *if and only if*

$$\langle \exp_z^{-1} x, \exp_z^{-1} y \rangle \leq 0, \quad \forall y \in Q.$$

We end this section with the following crucial results.

A sequence $\{x_n\}$ is said to br *Fejér monotone* w.r.t. Q if for all $x \in Q$ and $n \in \mathbb{N}$,

$$d(x_{n+1}, x) \leq d(x_n, x).$$

The following lemma provides some properties of Fejér monotonicity sequences that are useful for establishing convergence and linear convergence results.

Lemma 2.9. [30] *Let* $\{x_n\} \subseteq M$ *be a Fejér monotone sequence w.r.t* Q. *Then the following conditions hold:*

(i) $\{x_n\}$ *is bounded, and,* $\lim_{n\to+\infty} x_n = x$ *if* x *is a cluster point of* $\{x_n\}$ *and* $x \in Q$.

(ii) *Let* $\alpha > 0$ *be such that*

$$\alpha d^2(x_n, Q) \leq d^2(x_n, Q) - d^2(x_{n+1}, Q), \quad \forall n \in \mathbb{N}. \tag{2.6}$$

Then $\{x_n\}$ *converges linearly to a point* x *in* Q*:*

$$d(x_n, x) \leq 2(\sqrt{1-\alpha})^n d(x_0, Q), \quad \forall n \in \mathbb{N}. \tag{2.7}$$

3. Two-step Cyclic Algorithm and Its Convergence

In this section, we introduce an iterative method for solving convex feasibility problems in the setting of Hadamard manifolds.

We first recall the concept of convex feasibility problem in Hadamard manifolds. Let $I := \{1, 2, \dots m\}$ and $\{C_i : i \in I\}$ be a family of nonempty closed geodesic convex subset of M. Then the problem is to find

$$x^* \in C := \bigcap_{i=1}^{m} C_i, \tag{3.1}$$

where C is assumed to be a nonempty set. Set $i_n := n \pmod m + 1$ and let $\{T_{i_n}^{(n)}\}$ be a family of firmly nonexpansive mappings from M to itself satisfying

$$F\left(T_{i_n}^{(n)}\right) \supseteq C_{i_n}, \quad \forall n \in \mathbb{N}.$$

The multi-step cyclic algorithm is defined as follows:

Algorithm 3.1. Let $x_0 \in M$ be an initial point and define a sequence $\{x_n\}$ by

$$\begin{cases} x_{n+1} = \exp_{x_n} \alpha_n \exp_{x_n}^{-1} T_{i_n}^{(n)}(y_n) \\ y_n = \exp_{x_n} \beta_n \exp_{x_n}^{-1} T_{i_n}^{(n)}(x_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.2}$$

where $\{\alpha_n\}, \{\beta_n\} \subseteq (0, 1)$ are *relaxation parameter sequences*.

The following lemma is required to prove our main convergent theorem.

Lemma 3.2. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 and $x \in C := \bigcap_{i=1}^{m} C_i$. Then the following assertions hold for all $n \in \mathbb{N}$*

(i)

$$d^2(x_{n+1}, x_n) \leq \frac{\alpha_n}{1-\alpha_n}(d^2(x_n, x) - d^2(x_{n+1}, x)). \tag{3.3}$$

(ii) The sequence $\{x_n\}$ is Fejér monotone w.r.t C. If furthermore

$$\liminf_{n\to+\infty} \alpha_n(1-\alpha_n) > 0, \tag{3.4}$$

then

$$\lim_{n\to+\infty} d(x_{n+1}, x_n) = 0, \tag{3.5}$$

and

$$\liminf_{n\to+\infty} \alpha_n\beta_n(1-\beta_n) > 0, \tag{3.6}$$

then

$$\lim_{n\to+\infty} d(x_n, y_n) = 0. \tag{3.7}$$

Proof. (i) Fix $n \in \mathbb{N}$, let $x \in C$ and $\gamma_n : [0,1] \to M$ be geodesic joining x_n to $T_{i_n}^{(n)}(x_n)$. Thus, (3.2) can be written as $y_n = \gamma_n(\beta_n)$. By using geodesic convexity of Riemannian distance, we get

$$\begin{aligned} d(y_n, x) &= d(\gamma_n(\beta_n), x) \\ &\leq (1-\beta_n)d(x_n, x) + \beta_n d\left(T_{i_n}^{(n)}(x_n), T_{i_n}^{(n)}(x)\right) \\ &\leq (1-\beta_n)d(x_n, x) + \beta_n d(x_n, x) \\ &= d(x_n, x). \end{aligned} \tag{3.8}$$

Let $\triangle\left(x, x_n, T_{i_n}^{(n)}(y_n)\right) \subseteq M$ be a geodesic triangle with vertices x, x_n and $T_{i_n}^{(n)}(y_n)$, and $\triangle\left(\overline{x}, \overline{x_n}, \overline{T_{i_n}^{(n)}(x_n)}\right) \subseteq \mathbb{R}^2$ be the corresponding comparison triangle. Then, we have

$$\begin{aligned} &d(x, x_n) = \|\overline{x} - \overline{x_n}\|, d\left(x_n, T_{i_n}^{(n)}(y_n)\right) = \left\|\overline{x_n} - \overline{T_{i_n}^{(n)}(y_n)}\right\|, \text{and} \\ &d\left(T_{i_n}^{(n)}(y_n), x\right) = \left\|\overline{T_{i_n}^{(n)}(y_n)} - \overline{x}\right\|. \end{aligned} \tag{3.9}$$

Recall from (3.2) that $x_{n+1} = \exp_{x_n} \alpha_n \exp_{x_n}^{-1} T_{i_n}^{(n)}(y_n)$, then

$$\overline{x_{n+1}} = (1-\alpha_n)\overline{x_n} + \alpha_n \overline{T_{i_n}^{(n)}(y_n)}.$$

In view of (2.5), we have

$$d(x_{n+1}, x) \leq \|\overline{x_{n+1}} - \overline{x}\|. \tag{3.10}$$

From expression (3.9), yields

$$\begin{aligned} d^2(x_{n+1},x) &\le \|\overline{x_{n+1}}-\overline{x}\|^2 \\ &= \left\|(1-\alpha_n)\overline{x_n}+\alpha_n\overline{T^{(n)}_{i_n}(y_n)}-\overline{x}\right\|^2 \\ &= \left\|(1-\alpha_n)(\overline{x_n}-\overline{x})+\alpha_n\left(\overline{T^{(n)}_{i_n}(y_n)}-\overline{x}\right)\right\|^2 \\ &= (1-\alpha_n)\|\overline{x_n}-\overline{x}\|^2+\alpha_n\left\|\overline{T^{(n)}_{i_n}(y_n)}-\overline{x}\right\|^2-\alpha_n(1-\alpha_n)\left\|\overline{x_n}-\overline{T^{(n)}_{i_n}(y_n)}\right\|^2 \\ &= (1-\alpha_n)d^2(x_n,x)+\alpha_n d^2\left(T^{(n)}_{i_n}(y_n),x\right)-\alpha_n(1-\alpha_n)d^2\left(x_n,T^{(n)}_{i_n}(y_n)\right) \\ &\le (1-\alpha_n)d^2(x_n,x)+\alpha_n d^2(y_n,x)-\alpha_n(1-\alpha_n)d^2\left(x_n,T^{(n)}_{i_n}(y_n)\right). \end{aligned} \tag{3.11}$$

Since $x_{n+1}=\exp_{x_n}\alpha_n\exp^{-1}_{x_n}T^{(n)}_{i_n}(y_n)$, then

$$d(x_{n+1},x_n)=\alpha_n d\left(x_n,T^{(n)}_{i_n}(y_n)\right). \tag{3.12}$$

Substituting (3.8) and (3.12) into (3.11), we have

$$\begin{aligned} d^2(x_{n+1},x) &\le (1-\alpha_n)d^2(x_n,x)+\alpha_n d^2(x_n,x)-\frac{(1-\alpha_n)}{\alpha_n}d^2(x_{n+1},x_n) \\ &= d^2(x_n,x)-\frac{(1-\alpha_n)}{\alpha_n}d^2(x_{n+1},x_n), \end{aligned} \tag{3.13}$$

and we further have

$$d^2(x_{n+1},x_n)\le\frac{\alpha_n}{1-\alpha_n}(d^2(x_n,x)-d^2(x_{n+1},x)). \tag{3.14}$$

As a result, condition (i) holds.

(ii) From (3.14), we have

$$d^2(x_{n+1},x_n) \le \frac{\alpha_n}{1-\alpha_n}d^2(x_n,x)-\frac{\alpha_n}{1-\alpha_n}d^2(x_{n+1},x),$$

which implies that

$$\begin{aligned} \frac{\alpha_n}{1-\alpha_n}d^2(x_{n+1},x) &\le \frac{\alpha_n}{1-\alpha_n}d^2(x_n,x)-d^2(x_{n+1},x_n) \\ &\le \frac{\alpha_n}{1-\alpha_n}d^2(x_n,x). \end{aligned}$$

Thus, $d(x_{n+1},x)\le d(x_n,x)$ for all $n\in\mathbb{N}$, which means that $\{x_n\}$ is Fejér monotone w.r.t. C. Next, we show that $\lim_{n\to+\infty}d(x_{n+1},x_n)=0$. Suppose that (3.4) holds. Then there exists $N\in\mathbb{N}$ and $\epsilon>0$ such that $\alpha_n(1-\alpha_n)\ge\epsilon$ for each $n\ge N$. Furthermore, we can verify that

$$\frac{\alpha_n}{1-\alpha_n}\le\frac{1}{\epsilon},\quad \forall n\ge N.$$

From (3.14), we have

$$d^2(x_{n+1}, x_n) \leq \frac{1}{\epsilon}(d^2(x_n, x) - d^2(x_{n+1}, x)), \quad \forall n \geq N.$$

Since $\{x_n\}$ is a Fejér monotone w.r.t. C, it follows that $\lim_{n\to+\infty} d(x_n, x)$ exists. By letting $n \to \infty$ to the last inequality, we can have $\lim_{n\to+\infty} d(x_{n+1}, x_n) = 0$.

Assume that (3.6) holds. Now, we show that $\lim_{n\to+\infty} d(x_n, y_n) = 0$. Fix $n \in \mathbb{N}$, let $\triangle\left(x, x_n, T_{i_n}^{(n)}(x_n)\right) \subseteq M$ be a geodesic triangle with vertices x, x_n and $T_{i_n}^{(n)}(x_n)$, and let $\triangle\left(\overline{x}, \overline{x_n}, \overline{T_{i_n}^{(n)}(x_n)}\right) \subseteq \mathbb{R}^2$ be the corresponding comparison triangle. Then, we obtain

$$\begin{aligned} & d(x, x_n) = \|\overline{x} - \overline{x_n}\|, \quad d\left(x_n, T_{i_n}^{(n)}(x_n)\right) = \left\|\overline{x_n} - \overline{T_{i_n}^{(n)}(x_n)}\right\|, \text{ and} \\ & d\left(T_{i_n}^{(n)}(x_n), x\right) = \left\|\overline{T_{i_n}^{(n)}(x_n)} - \overline{x}\right\|. \end{aligned} \tag{3.15}$$

Recall from (3.2) that $y_n = \exp_{x_n} \beta_n \exp_{x_n}^{-1} T_{i_n}^{(n)}(x_n)$ and set

$$\overline{y_n} = (1 - \beta_n)\overline{x_n} + \beta_n \overline{T_{i_n}^{(n)}(x_n)}.$$

In view of (2.5) and (3.15), yields

$$\begin{aligned} d^2(y_n, x) &\leq \|\overline{y_n} - \overline{x}\|^2 \\ &= \left\|(1-\beta_n)\overline{x_n} + \beta_n \overline{T_{i_n}^{(n)}(x_n)} - \overline{x}\right\|^2 \\ &= \left\|(1-\beta_n)(\overline{x_n} - \overline{x}) + \beta_n\left(\overline{T_{i_n}^{(n)}(x_n)} - \overline{x}\right)\right\|^2 \\ &= (1-\beta_n)\|\overline{x_n} - \overline{x}\|^2 + \beta_n\left\|\overline{T_{i_n}^{(n)}(x_n)} - \overline{x}\right\|^2 - \beta_n(1-\beta_n)\left\|\overline{x_n} - \overline{T_{i_n}^{(n)}(x_n)}\right\|^2 \\ &= (1-\beta_n)d^2(x_n, x) + \beta_n d^2\left(T_{i_n}^{(n)}(x_n), x\right) - \beta_n(1-\beta_n)d^2\left(x_n, T_{i_n}^{(n)}(x_n)\right) \\ &\leq (1-\beta_n)d^2(x_n, x) + \beta_n d^2(x_n, x) - \beta_n(1-\beta_n)d^2\left(x_n, T_{i_n}^{(n)}(x_n)\right) \\ &= d^2(x_n, x) - \beta_n(1-\beta_n)d^2\left(x_n, T_{i_n}^{(n)}(x_n)\right). \end{aligned} \tag{3.16}$$

Since $y_n = \exp_{x_n} \beta_n \exp_{x_n}^{-1} T_{i_n}^{(n)}(x_n)$, we deduce that

$$d(x_n, y_n) = \beta_n d\left(x_n, T_{i_n}^{(n)}(x_n)\right). \tag{3.17}$$

Substitution (3.17) into (3.16), we get

$$d^2(y_n, x) \leq d^2(x_n, x) - \frac{(1-\beta_n)}{\beta_n} d^2(x_n, y_n). \tag{3.18}$$

By combining (3.11) and (3.18), we have

$$\begin{aligned} d^2(x_{n+1}, x) &\leq (1-\alpha_n)d^2(x_n, x) + \alpha_n d^2(y_n, x) \\ &\leq (1-\alpha_n)d^2(x_n, x) + \alpha_n\left(d^2(x_n, x) - \frac{(1-\beta_n)}{\beta_n}d^2(x_n, y_n)\right) \\ &= d^2(x_n, x) - \frac{\alpha_n(1-\beta_n)}{\beta_n}d^2(x_n, y_n), \end{aligned}$$

and we further have

$$d^2(x_n, y_n) \leq \frac{\beta_n}{\alpha_n(1-\beta_n)}(d^2(x_n, x) - d^2(x_{n+1}, x)). \tag{3.19}$$

Since (3.6) holds, then there exists $N \in \mathbb{N}$ and $\eta > 0$ such that $\alpha_n\beta_n(1-\beta_n) \geq \eta$ for all $n \geq N$. It is easy to check that

$$\frac{\beta_n}{\alpha_n(1-\beta_n)} \leq \frac{1}{\eta}, \quad \forall n \geq N.$$

From (3.19), implies that

$$d^2(x_n, y_n) \leq \frac{1}{\eta}(d^2(x_n, x) - d^2(x_{n+1}, x)), \quad \forall n \geq N.$$

Recall that $\{x_n\}$ is Fejér monotone w.r.t. C so that $\lim_{n\to+\infty} d(x_n, x)$ exists. Hence, from the above inequality, we have $\lim_{n\to+\infty} d(x_n, y_n) = 0$. The proof is therefore completed. ∎

Following the definitions of focusing algorithm and linearly focusing algorithm, we will prove that the sequence $\{x_n\}$ generated by Algorithm 3.1 converges to a point in C.

Definition 3.3. [4] An algorithm is said to be

(i) *focusing* if for all $j \in I$ and every subsequence $\{x_{n_k}\}$ of $\{x_n\}$,

$$\left.\begin{array}{l} x_{n_k} \to x \\ d(x_{n_k}, T_j^{(n_k)}(x_{n_k})) \to 0 \\ i_{n_k} = j \quad \text{for all } k \in \mathbb{N}. \end{array}\right\} \Longrightarrow x \in C_j; \tag{3.20}$$

(ii) *linearly focusing* if there is $\lambda > 0$ such that

$$\lambda d(x_n, C_{i_n}) \leq d(x_n, C_{i_n}^{(n)}) \quad \text{for all } n \in \mathbb{N}, \tag{3.21}$$

where $\{x_n\}$ is a sequence generated by Algorithm 3.1 and $C_{i_n}^{(n)}$ is a closed geodesic convex nonempty set containing C_{i_n}.

Every a linearly focusing algorithm is a focusing algorithm.

Remark 3.4. [30] In the case when the sequence $\left\{T_{i_n}^{(n)}\right\}$ of firmly nonexpansive mappings satisfies that $F(T_{i_n}) = C_{i_n}$ and $T_{i_n}^{(n)} = T_{i_n}$ for all $n \in \mathbb{N}$, the algorithm is linearly focusing; in particular the algorithm is linearly focusing when $T_{i_n}^{(n)} = P_{C_{i_n}}$ for all $n \in \mathbb{N}$.

We can now present the main result as follows.

Theorem 3.5. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1. Assume that Algorithm 3.1 is focusing and $\{\alpha_n\}$, $\{\beta_n\}$ satisfy (3.4), (3.6), respectively. Then the sequence $\{x_n\}$ converges to a point in C.*

Proof. As a consequence of Lemma 3.2, $\{x_n\}$ is Fejér monotone w.r.t C and is bounded by Lemma 2.9. Thus there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to+\infty} x_{n_k} = x^* \in M$. We shall verify $x^* \in C$ in this proof.

Noting the definition of $\{i_{n_k}\}$, without loss of generality we suppose that $i_{n_k} = m$ for all k. Let $j \in I$ and consider the subsequence $\{x_{n_k+j}\}_{k=0}^{\infty}$. From (3.5), we have

$$\lim_{k\to\infty} x_{n_k+j} = x^* \quad \text{and} \quad i_{n_k+j} = j. \tag{3.22}$$

Because $\{\beta_n\} \subseteq (0,1)$ then $\{\beta_n\}$ is bounded below by some positive numbers. In view of (3.17) and the fact that $\lim_{n\to+\infty} d(x_n, y_n) = 0$, we get

$$\begin{aligned} \lim_{k\to+\infty} d\left(x_{n_k+j}, T_j^{(n_k+j)}(x_{n_k+j})\right) &= \lim_{k\to+\infty} \frac{1}{\beta_{n_k}} d\left(x_{n_k+j}, y_{n_k+j}\right) \\ &= 0. \end{aligned}$$

This, together with (3.22), yields that $x^* \in C_j$ since Algorithm 3.1 is focusing. Therefore, $x^* \in C$ as $j \in I$ arbitrary. Moreover, by Lemma 2.9, $\lim_{n\to+\infty} x_n = x^*$ as required. The proof is therefore completed.

∎

According ot Remark 3.4, we can have the following corollary.

Corollary 3.6. *Suppose that* $T_{i_n}^{(n)} :\equiv P_{C\,i_n}$ *and* $\{\alpha_n\}$, $\{\beta_n\}$ *satisfy condition* (3.4) *and* (3.6), *respectively. Then any sequence* $\{x_n\}$ *generated by Algorithm 3.1 converges to a point in* C.

Proof. From Proposition 2.8, P_{C_i} is firmly nonexpansive for any $i \in I$. Furthermore, Algorithm 3.1 is focusing by Remark 3.4. Follows from the proof of Theorem 3.5, and is thus omitted. ∎

4. Linear convergence of Two-step Cyclic Projection Algorithm

In this section, we discuss the linear convergence of the Algorithm 3.1 where each $T_{i_n}^{(n)}$ is the projection onto some closed convex nonempty set $C_{i_n}^{(n)}$ containing C_{i_n}, i.e.,

$$T_{i_n}^{(n)} := P_{C_{i_n}^{(n)}} \text{ and } C_{i_n}^{(n)} \supseteq C_{i_n}, \quad \forall n \in \mathbb{N}. \tag{4.1}$$

Next, let us present the concept of linear convergence.

Definition 4.1. Let $\{x_n\} \subset M$ such that $\{x_n\}$ converges to a point $x \in M$. Then, the convergence is said to be *linear convergence* if and only if there exist a constant $\theta < 1$ and a positive $N \in \mathbb{N}$ such that

$$d(x_n, \overline{x}) \leq \theta d(x_{n-1}, x), \quad \forall n > N.$$

For establishing the linear convergence results, we need the definitions of linear regularity and bounded linear regularity.

Definition 4.2. [30] A family $\{C_i : i \in I\}$ is called

(i) *linearly regular* if there exists $\tau > 0$ such that

$$d(x, C) \leq \tau \max_{i \in I}\{d(x, C_i)\}, \tag{4.2}$$

for all $x \in M$.

(ii) *bounded linearly regular* if, for any bounded subset $S \subseteq M$, there exist $\tau_S > 0$ such that (4.2) holds for any $x \in S$ with $\tau = \tau_S$.

Next, we present and prove the linear convergence theorem for the two-step cyclic projection algorithm.

Theorem 4.3. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1. Suppose that Algorithm 3.1 is linearly focusing, the family $\{C_i : i \in I\}$ is bounded linearly regular, and the conditions (3.4) and (3.6) hold. Then $\{x_n\}$ converges linearly to a point $x \in C$.*

Proof. Lemma 3.2 and Theorem 3.5 are applicable according to the assumptions. Then, $\{x_n\}$ is Fejér monotone w.r.t. C and converges to a point $x \in C$. Next, we show that subsequence $\{x_{km}\}_{k=0}^{+\infty}$ converges linearly, i.e. there is some $\gamma > 0$ and $\theta \in (0,1)$ such that

$$d(x_{km}, x) \leq \gamma\theta^k, \quad \forall k \in \mathbb{N}. \tag{4.3}$$

To verify (4.3), without loss of generality, since (3.4) holds, then there exists $\epsilon > 0$ such that

$$\alpha_n(1-\alpha_n) \geq \epsilon > 0, \quad \forall n \in \mathbb{N}. \tag{4.4}$$

This implies that

$$\frac{\alpha_n}{1-\alpha_n} \leq \frac{1}{\epsilon}, \quad \forall n \in \mathbb{N}. \tag{4.5}$$

Let $i \in I$ and $k \in \mathbb{N}$. It is easy to see that

$$d(x_{km}, x_{km+i}) \leq \sum_{j=0}^{m-1} d(x_{km+j}, x_{km+j+1}).$$

From the last inequality, we get

$$\begin{aligned} d^2(x_{km}, x_{km+i}) &\leq \left(\sum_{j=0}^{m-1} 1 \cdot d(x_{km+j}, x_{km+j+1})\right)^2 \\ &\leq \left(\sum_{j=0}^{m-1} 1^2\right)\left(\sum_{j=0}^{m-1} d^2(x_{km+j}, x_{km+j+1})\right) \\ &= m\sum_{j=0}^{m-1} d^2(x_{km+j}, x_{km+j+1}). \end{aligned}$$

Substitution (3.3) into the last inequality, we obtain

$$d^2(x_{km}, x_{km+i}) \leq m\sum_{j=0}^{m-1} \frac{\alpha_{km+j}}{1-\alpha_{km+j}}(d^2(x_{km+j}, z) - d^2(x_{km+j+1}, z))$$

for any $z \in C$, and it follows from (4.5),

$$d^2(x_{km}, x_{km+i}) \leq \frac{m}{\epsilon}(d^2(x_{km}, C) - d^2(x_{(k+1)m}, C)). \tag{4.6}$$

Similarly, from (3.6) holds, then there exist $\eta > 0$ such that

$$\alpha_n\beta_n(1-\beta_n) \geq \eta > 0, \quad \forall n \in \mathbb{N}. \tag{4.7}$$

This implies that

$$\frac{\beta_n}{\alpha_n(1-\beta_n)} \leq \frac{1}{\eta}, \quad \forall n \in \mathbb{N}. \tag{4.8}$$

We also have

$$\frac{1}{\alpha_n\beta_n(1-\beta_n)} \leq \frac{1}{\eta} \Longrightarrow \frac{1}{\beta_n} \leq \frac{\alpha_n(1-\beta_n)}{\eta} \leq \frac{1}{\eta}, \quad \forall n \in \mathbb{N}. \tag{4.9}$$

For $i \in I$ and $k \in \mathbb{N}$. It easy to see that

$$d(x_{km+i}, y_{km+i}) \leq \sum_{j=1}^{m} d(x_{km+j}, y_{km+j}).$$

Following from the above inequality, we obtain

$$\begin{aligned} d^2(x_{km+i}, y_{km+i}) &\leq \left(\sum_{j=1}^{m} 1 \cdot d(x_{km+j}, y_{km+j})\right)^2 \\ &\leq \left(\sum_{j=1}^{m} 1^2\right)\left(\sum_{j=0}^{m} d^2(x_{km+j}, y_{km+j})\right) \\ &= m\sum_{j=1}^{m} d^2(x_{km+j}, y_{km+j}). \end{aligned}$$

In view of (3.19), we conclude that

$$d^2(x_{km+i}, y_{km+i}) \leq m\sum_{j=1}^{m} \frac{\beta_{km+j}}{\alpha_{km+j}(1-\beta_{km+j})}(d^2(x_{km+j}, z) - d^2(x_{km+j+1}, z))$$

for any $z \in C$. Summing up the last inequality and applying (4.8), we get

$$d^2(x_{km+i}, y_{km+i}) \leq \frac{m}{\eta}(d^2(x_{km}, C) - d^2(x_{(k+1)m}, C)). \tag{4.10}$$

From algorithm is linearly focusing, there exists $\lambda > 0$ (independent of i and k) such that

$$\lambda d(x_{km+i}, C_i) \leq d\left(x_{km+i}, C_i^{(km+i)}\right). \tag{4.11}$$

In view of (3.2) and (4.1), we get

$$\begin{aligned} d(x_{km+i}, C_i^{(km+i)}) &= d\left(x_{km+i}, T_i^{(km+i)}(x_{km+i})\right) \\ &= \frac{1}{\beta_{km+i}} d(x_{km+i}, y_{km+i}). \end{aligned} \tag{4.12}$$

Substitution (4.9) and (4.10) into the last inequality,

$$\begin{aligned} d^2(x_{km+i}, C_i) &\leq \frac{1}{\lambda^2 \beta_{km+i}^2}\left[\frac{m}{\eta}(d^2(x_{km}, C) - d^2(x_{(k+1)m}, C))\right] \\ &\leq \frac{m}{\lambda^2\eta^3}\left(d^2(x_{km}, C) - d^2(x_{(k+1)m}, C)\right). \end{aligned} \tag{4.13}$$

Consider,

$$\begin{aligned} d^2(x_{km}, C_i) &\leq (d(x_{km}, x_{km+i}) + d(x_{km+i}, C_i))^2 \\ &\leq 2d^2(x_{km}, x_{km+i}) + 2d^2(x_{km+i}, C_i). \end{aligned} \tag{4.14}$$

By combing (4.6), (4.13) and (4.14), we obtain

$$d^2(x_{km}, C_i) \leq \left(\frac{2m}{\epsilon} + \frac{2m}{\lambda^2\eta^3}\right)(d^2(x_{km}, C) - d^2(x_{(k+1)m}, C)). \tag{4.15}$$

From the fact that the family $\{C_i : i \in I\}$ is bounded linearly regular and $\{x_n\}$ is bounded, then there exists $\tau > 0$ such that

$$d(x_n, C) \leq \tau \max_{i\in I}\{d(x_n, C_i)\}, \quad \forall\ n \in \mathbb{N}.$$

Thereby,

$$\begin{aligned} d^2(x_{km}, C) &\leq \tau^2 \max_{i\in I}\{d^2(x_{km}, C_i)\} \\ &\leq \tau^2\left(\frac{2m}{\epsilon} + \frac{2m}{\lambda^2\eta^3}\right)(d^2(x_{km}, C) - d^2(x_{(k+1)m}, C)). \end{aligned}$$

The subsequence $\{x_{km}\}_{k=0}^{+\infty}$ is linearly converges to $x \in C$ by using (ii) of Lemma 2.9. This implies that (4.3) holds. Fix $n \in \mathbb{N}$, and set

$$n = km + r \quad \text{where } r \in \{0, 1, \dots, m-1\}.$$

Then we conclude

$$d(x_n, x) \leq d(x_{km}, x) \leq \gamma(\theta^{\frac{1}{m}})^{km} = \frac{\gamma(\theta^{\frac{1}{m}})^{km+r}}{\theta^{\frac{r}{m}}} \leq \frac{\gamma}{\theta}(\theta^{\frac{1}{m}})^n,$$

and complete the proof. ∎

We obtain the following corollary from Remark 3.4 and Theorem 4.3 in the spacial case when $C_{i_n}^{(n)} = C_{i_n}$ in (4.1) for all $n \in \mathbb{N}$.

Corollary 4.4. *Let $\{x_n\}$ be a sequence generated by the two-step cyclic projection algorithm. Suppose that conditions (3.4), (3.6) hold, the family $\{C_i : i \in I\}$ is boundedly linearly regular. Suppose further that, for all $n \in \mathbb{N}$, $C_{i_n}^{(n)} = C_{in}$ in (4.1). Then $\{x_n\}$ converges linearly to a point in C.*

5. Numerical Example

In this section, we provide a numerical examples in Hadamard manifolds to illustrate the convergence behavior of Algorithms 3.1. All the programs are written in `Matlab R2016b` and computed on `PC Intel(R) Core(TM) i7 @1.80 GHz` and a `8 GB 1600 MHz DDR3` Memory.

Let $M = \mathbb{H} := \{(t_1, t_2) \in \mathbb{R}^2 \mid t_2 > 0\}$ be the *Poincaré plane* endowed with the Riemannian metric defined by

$$g_{11} = g_{22} := \frac{1}{t_2^2}, g_{12} := 0 \text{ for any } (t_1, t_2) \in \mathbb{H}. \tag{5.1}$$

The sectional curvature of $\mathbb{H}$ is equal to -1 and the geodesics of the Poincaré plane are the semilines $\gamma_a : t_1 = a, t_2 > 0$ and the semicircles $\gamma_{b,r} : (t_1 - b)^2 + t_2^2 = r^2, t_2 > 0$; or admit the following natural parameterizations

$$\begin{aligned} &\gamma_a : t_1 = a,\ \ t_2 = e^s, && s \in (-\infty, +\infty);\\ &\gamma_{b,r} : t_1 = -r\tanh s,\ \ t_2 = \frac{r}{\cosh s}, && s \in (-\infty, +\infty); \end{aligned} \tag{5.2}$$

see e.g., [29]. Furthermore, consider two points $y = (t_1^y, t_2^y)$ and $z = (t_1^z, t_2^z)$ in $\mathbb{H}$. Then the Riemannian distance between y, z is given by

$$d_{\mathbb{H}}(y, z) = \begin{cases} \left|\ln \dfrac{t_2^z}{t_2^y}\right|, & \text{if } t_1^y = t_1^z,\\ \left|\ln \dfrac{t_1^y - b + r}{t_1^z - b + r} \cdot \dfrac{t_2^z}{t_2^y}\right|, & \text{if } t_1^y \neq t_1^z, \end{cases}$$

where

$$b = \frac{(t_1^y)^2 + (t_2^y)^2 - ((t_1^z)^2 + (t_2^z)^2)}{2(t_1^y - t_1^z)} \quad \text{and} \quad r = \sqrt{(t_1^y - b)^2 + (t_2^y)^2}.$$

To get the expression of $\exp_y^{-1} z$, we consider a smooth geodesic curve γ joining y to z defined by

$$\gamma(s) := (\gamma_1(s), \gamma_2(s)), \quad s \in [0, 1]$$

where for each $s \in [0, 1]$, $\gamma_1(s)$ and $\gamma_2(s)$ are respectively defined by

$$\gamma_1(s) := \begin{cases} t_1^y, & \text{if } t_1^y = t_1^z,\\ b - r\tanh\left((1 - s) \cdot \operatorname{arctanh}\dfrac{b - t_1^y}{r} + s \cdot \operatorname{arctanh}\dfrac{b - t_1^z}{r}\right), & \text{if } t_1^y \neq t_1^z, \end{cases}$$

and

$$\gamma_2(s) := \begin{cases} e^{(1-s)\cdot \ln t_2^y + s \cdot \ln t_2^z}, & \text{if } t_1^y = t_1^z,\\ \dfrac{r}{\cosh\left((1 - s) \cdot \operatorname{arctanh}\dfrac{b - t_1^y}{r} + s \cdot \operatorname{arctanh}\dfrac{b - t_1^z}{r}\right)}, & \text{if } t_1^y \neq t_1^z. \end{cases}$$

By the Riemannian metric endowed on $\mathbb{H}$ (c.f. (5.1)), one checks that

$$\gamma'(0) = \left(\frac{d\gamma_1(s)}{ds}, \frac{d\gamma_2(s)}{ds}\right)\bigg|_{s=0};$$

see [14], page 7. Therefore, by elementary calculus, we get that

$$\exp_y^{-1} z = \gamma^{'}(0) = \begin{cases} \left(0, t_2^y \ln \frac{t_2^z}{t_2^y}\right), & \text{if } t_1^y = t_1^z, \\ \frac{t_2^y \left(\operatorname{arctanh} \frac{b - t_1^y}{r} - \operatorname{arctanh} \frac{b - t_1^z}{r}\right)}{r} (t_2^y, b - t_1^y), & \text{if } t_1^y \neq t_1^z. \end{cases} \tag{5.3}$$

Example 5.1. Follows from [30], the Example 5.1: Let $M = \mathbb{H}^2$ and C_1, C_2 be closed convex subsets of M defined as

$$C_1 := \{(t_1, t_2) \in M : t_2 \geq 1\}$$

and

$$C_2 := \{(t_1, t_2 \in M : t_1^2 + t_2^2 \leq 1)\}.$$

From [29], page 301, C_1 and C_2 are convex because that C_1 is level set convex function $f : M \to \mathbb{R}$ defined by

$$f(y) = \frac{1}{t_2}, \ \forall y \in (t_1, t_2) \in M$$

and $\gamma := \{(t_1, t_2) \in M : t_1 + t_2 = 1\}$ is a geodesic of M, receptively. Moreover $C = C_1 \cap C_2 = \{(0, 1)\}$,

$$P_{C_1}(x) = (t_1, 1), \ \forall x = (t_1, t_2) \notin C_1$$

and

$$P_{C_2}(x) = \left(\frac{2t_1}{t_1^2 + t_2^2 + 1}, \sqrt{1 - \left(\frac{2t_1}{t_1^2 + t_2^2 + 1}\right)}\right) \text{ for any } x = (t_1, t_2) \notin C_2.$$

We implement the projection algorithm to find $x^* = (0, 1) \in C_1 \cap C_2$ which is defined as: Choose $x_0 \in M$ and defined x_{n+1} by

$$\begin{cases} x_{n+1} = \exp_{x_n} \alpha_n \exp_{x_n}^{-1} T_{i_n}^{(n)}(y_n) \\ y_n = \exp_{x_n} \beta_n \exp_{x_n}^{-1} T_{i_n}^{(n)}(x_n), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $i_n := (n \mod 2) + 1$ and $\{\alpha_n\}, \{\beta_n\} \in (0, 1)$ are constants. The numerical results are listed in Table 1 and Table 2 with an initial point $x_0 = (1, 1)$ and $x_0 = (3, 2)$, respectively; and each one shows the results for three different constant relaxation parameters, $\alpha_n = 0.3, 0.6, 0.9$ and $\beta_n = 0.3, 0.6, 0.9$, respectively. Moreover, the numerical results displayed on Figure 1 and Figure 2 which depicts the " Distance to Solution" versus "Iteration Number". The numerical results show the convergence tendency of the algorithm as predicted by Theorem 3.5; furthermore, we observe that the bigger the relaxation parameters α_n and β_n, the faster the algorithm converges.

Table 1. The comparison of Projection Algorithm for the initial point is $x_0 = (1, 1)$ with relaxation parameters α_n and β_n.

Iteration No.	Initial point $x_0 = (1, 1)$		
	$\alpha_n = 0.3, \beta_n = 0.3$	$\alpha_n = 0.6, \beta_n = 0.6$	$\alpha_n = 0.9, \beta_n = 0.9$
1	(0.962023, 0.980092)	(0.857514, 0.914993)	(0.715276, 0.796372)
2	(0.962023, 0.981867)	(0.857514, 0.944729)	(0.715276, 0.957661)
3	(0.928696, 0.962795)	(0.773302, 0.878510)	(0.609786, 0.835884)
4	(0.928696, 0.966086)	(0.773302, 0.920445)	(0.609786, 0.966513)
5	(0.899310, 0.947932)	(0.716982, 0.867794)	(0.542662, 0.870441)
6	(0.899310, 0.952505)	(0.716982, 0.913244)	(0.542662, 0.973981)
7	(0.873262, 0.935291)	(0.675205, 0.869113)	(0.494071, 0.893398)
8	(0.873262, 0.940939)	(0.675205, 0.914132)	(0.494071, 0.978810)
9	(0.850044, 0.924649)	(0.641820, 0.875387)	(0.456705, 0.909423)
10	(0.850044, 0.931191)	(0.64182, 0.918349)	(0.456705, 0.982122)
⋮	⋮	⋮	⋮
50	(0.630933, 0.904178)	(0.382564, 0.970729)	(0.232119, 0.995622)
⋮	⋮	⋮	⋮
100	(0.520102, 0.933159)	(0.284848, 0.983952)	(0.167478, 0.997740)

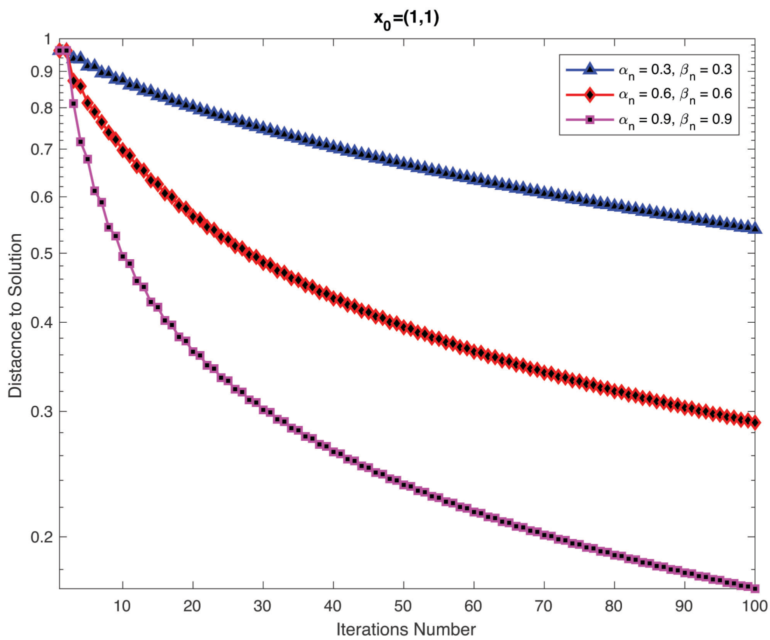

Fig. 1. Distance to solution $x^* = (0, 1)$ of each iteration number where the initial point is $x_0 = (1, 1)$.

Table 2. The comparison of Projection Algorithm for the initial point is $x_0 = (3, 2)$ with relaxation parameters α_n and β_n.

Iteration No.	Initial point $x_0 = (3, 2)$		
	$\alpha_n = 0.3, \beta_n = 0.3$	$\alpha_n = 0.6, \beta_n = 0.6$	$\alpha_n = 0.9, \beta_n = 0.9$
1	(2.694926, 1.972871)	(1.842587, 1.799538)	(0.784290, 1.157324)
2	(2.694926, 1.855837)	(1.842587, 1.456475)	(0.784290, 1.028150)
3	(2.413940, 1.912864)	(1.227856, 1.354538)	(0.617482, 0.850046)
4	(2.413940, 1.804398)	(1.227856, 1.214357)	(0.617482, 0.969603)
5	(2.162268, 1.832029)	(0.947803, 1.084283)	(0.546490, 0.869407)
6	(2.162268, 1.734876)	(0.947803, 1.053153)	(0.947803, 1.053153)
7	(1.941625, 1.740411)	(0.811160, 0.957026)	(0.496873, 0.892178)
8	(1.941625, 1.655744)	(0.811160, 0.972280)	(0.496873, 0.978556)
9	(1.751247, 1.645610)	(0.733998, 0.903834)	(0.458905, 0.908521)
10	(1.751247, 1.573466)	(0.733998, 0.937339)	(0.458905, 0.981937)
⋮	⋮	⋮	⋮
50	(0.708503, 0.918746)	(0.395949, 0.968588)	(0.232401, 0.995611)
⋮	⋮	⋮	⋮
100	(0.552972, 0.924655)	(0.290245, 0.983330)	(0.167584, 0.997737)

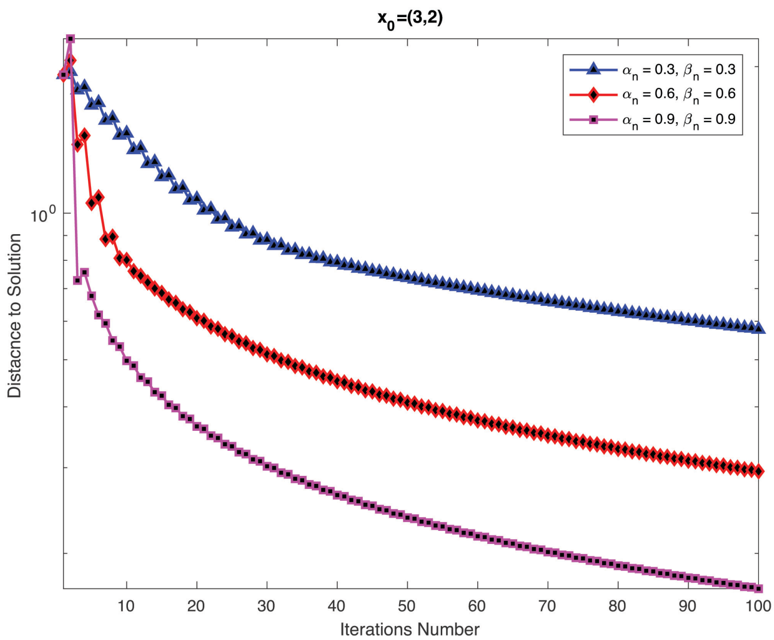

Fig. 2. Distance to solution $x^* = (0, 1)$ of each iteration number where the initial point is $x_0 = (2, 3)$.

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A Family of Conjugate Gradient Projection Method for Nonlinear Monotone Equations with Applications to Compressive Sensing

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ABSTRACT

In this work, we propose a family of conjugate gradient projection method for nonlinear monotone equations with convex constraints. Under some appropriate assumptions, the global convergence of the method is established. Numerical examples reported shows that the method is competitive and efficient for solving monotone nonlinear equations. Furthermore, we apply the proposed algorithm to solve the sparse signal reconstruction problem in compressive sensing.

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1. Introduction

Consider finding a point $x \in \Omega$ such that

$$F(x) = 0, \tag{1.1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and monotone, that is, $\langle F(x)-F(y), (x-y)\rangle \geq 0, \quad \forall x, y \in \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ is nonempty and convex. The corresponding unconstrained problem when $\Omega = \mathbb{R}^n$ have been discussed extensively, and many iterative methods have been proposed by many researchers. Some examples are; Newton method, quasi-Newton method, Gauss-Newton, Levenberg-Marquardt method and their variants (see[1, 5, 6, 17, 7, 9, 14, 15, 16, 19, 21, 22, 24, 27, 28]). With a good initial guess, these algorithms are very attractive as they have fast convergence rate. However, there are relatively scanty literatures on constrained problem (1.1).

Constrained problem (1.1) has so many practical applications, for example in chemical equilibrium systems and economic equilibrium problems (see[20, 8]). Iterative methods for solving constrained monotone nonlinear equations have recently receive relatively high attention [18, 26, 34, 30, 32, 36, 33]. For example, in [26] Wang et al. proposed a projection method which requires no differentiability and regularity conditions for solving (1.1). Numerical experiments presented in the paper indicates the efficiency of the method. Ma and Wang [18] proposed a modified extragradient method for solving constrained monotone equations. A spectral gradient approach and a projection technique was presented by Yu et al. [33] for convex constrained problems. Using similar projection technique approach, Zheng [36] proposed a spectral gradient method for constrained problems. Also, Yu et al. in [32] proposed a multivariate spectral gradient

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projection (SGP) for solving problems of the form (1.1). A remarkable property of these gradient-type algorithms is that the direction does not depend on the gradient information, therefore can be applied to solve nonsmooth equations. However, Xiao and Zhu [30] proposed a projected conjugate gradient (CGD) to solve constrained problems. This method can be viewed as an extension of the CG−Descent method for solving convex constrained problems.

Motivated by these methods, we propose a family of conjugate gardient projection method for constrained nonlinear monotone equations, which is an extension of the method of Feng et al. [10] for solving convex constrained problems. The method possesses some properties, which are; (1) the method is derivative-free which implies its applicability in handling nonsmooth equations; (2) the global convergence was established without differentiability assummption and (3) it is independent of any merit function.

The remaining part of the paper is organized as follows. Section 2 provides the proposed method and its algorithm. Section 3 gives the global convergence and in Section 4 we report numerical results to show its practical performance, and apply it to solve the sparse signal reconstruction in compressive sensing.

2. Preliminaries and algorithm

In this section, we first give some basic concepts and properties. Let Ω be a nonempty closed convex subset of $\mathbb{R}^n$. Then for all $x \in \mathbb{R}^n$, its projection onto Ω is defined as

$$P_\Omega(x) = \arg\min\{\|x - y\| \ : y \in \Omega\}.$$

The map $P_\Omega : \mathbb{R}^n \to \Omega$ is called a projection operator and has the nonexpansive property, that is, for all $x, y \in \mathbb{R}^n$,

$$\|P_\Omega(x) - P_\Omega(y)\| \leq \|x - y\| \qquad \forall x, y \in \mathbb{R}^n. \tag{2.1}$$

The following propositions [31, 35] give some basic properties of the projection operator P_Ω.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be nonempty, closed and convex. Then for all $x \in \mathbb{R}^n$ and $y \in \Omega$,*

$$(P_\Omega(x) - x)^T(y - P_\Omega(x)) \geq 0.$$

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^n$ be nonempty, closed and convex. Then for all $x, d \in \mathbb{R}^n$ and $\alpha \geq 0$, define $x(\alpha) := P_\Omega(x - \alpha d)$. Then $d^T(x(\alpha) - x)$ is nonincreasing with respect to $\alpha \geq 0$.*

The following assumptions hold throughout this paper.

Assumption A (i) The solution set of problem (1.1) is nonempty. (ii) The function F is Lipschitz continuous, that is there exists a positive constant L such that

$$\|F(x) - F(y)\| \leq L\|x - y\|, \tag{2.2}$$

for all $x, y \in \mathbb{R}^n$.

Assumption (ii) implies there is a positive constant τ such that

$$\|F(x_k)\| \leq \tau \quad \forall k \geq 0. \tag{2.3}$$

Now all is set to describe our proposed algorithm, which is an extension of the method in [10] to solve convex constrained problems.

Algorithm 1: Family of Conjugate Gradient Projection Method (FCG)

Step 0. Given an arbitrary initial point $x_0 \in \mathbb{R}^n$, parameters $0 < r < 1$, $\eta \geq 0$, $\sigma > 0$, $t > 0$, $\rho > 0$, $\epsilon > 0$, and set $k := 0$.

Step 1. If $\|F(x_k)\| \leq \epsilon$, stop, otherwise go to **Step 2**.

Step 2. Compute

$$d_k = \begin{cases} -F(x_k), & \text{if } k = 0, \\ -\left(1 + \beta_k \frac{F(x_k)^T d_{k-1}}{\|F(x_k)\|^2}\right) F(x_k) + \beta_k d_{k-1}, & \text{if } k \geq 1, \end{cases} \tag{2.4}$$

where β_k is such that

$$|\beta_k| \leq t \frac{\|F(x_k)\|}{\|d_{k-1}\|}, \quad \forall k \geq 1, \quad t > 0. \tag{2.5}$$

Step 3. Find the trial point $y_k = x_k + \alpha_k d_k$, where $\alpha_k = \rho r^{m_k}$ and m_k is the smallest nonnegative integer m such that

$$-\langle F(x_k + \rho r^m d_k), d_k \rangle \geq \sigma \rho r^m \|d_k\|. \tag{2.6}$$

Step 4. If $y_k \in \Omega$ and $\|F(y_k)\| \leq \epsilon$, stop. Else compute the next iterate

$$x_{k+1} = P_\Omega[x_k - \zeta_k F(y_k)]$$

where

$$\zeta_k = \frac{F(y_k)^T (x_k - y_k)}{\|F(y_k)\|^2}.$$

Step 5. Let $k = k + 1$ and go to **Step 1**.

Remark 2.3. From the definition of d_k, we have

$$\langle F(x_k), d_k \rangle = -F(x_k)^T F(x_k) - \frac{\beta_k F(x_k)^T F(x_k) F(x_k)^T d_{k-1}}{\|F(x_k)\|^2} + \beta_k F(x_k)^T d_{k-1} = -\|F(x_k)\|^2 \tag{2.7}$$

which means the direction d_k is sufficiently descent.

Remark 2.4. Remark 2.3 together with the Cauchy-Schwartz inequality implies that $\|d_k\| \geq \|F(x_k)\|$. Furthermore, by (2.4) and (2.5), we get

$$\begin{aligned} \|d_k\| &\leq \|F(x_k)\| + |\beta_k| \frac{\|F(x_k)\| \|d_{k-1}\|}{\|F(x_k)\|^2} \|F(x_k)\| + |\beta_k| \|d_{k-1}\| \\ &\leq \|F(x_k)\| + t\|F(x_k)\| + t\|F(x_k)\| \\ &\leq (1 + 2t)\|F(x_k)\|. \end{aligned}$$

Therefore,

$$\|F(x_k)\| \leq \|d_k\| \leq (1 + 2t)\|F(x_k)\|, \quad \forall k \geq 0, \tag{2.8}$$

which implies boundedness of the search direction.

3. Convergence analysis

To prove the global convergence of **Algorithm 1**, the following lemmas are needed. The following lemma shows that **Algorithm 1** is well-defined.

Lemma 3.1. *Suppose F is continuous, monotone and **Assumption A** (i) hold, then there exists a step-length α_k satisfying the line search (2.6) $\forall k \geq 0$.*

Proof. Suppose there exists $k_0 \geq 0$ such that (2.6) does not hold for any nonnegative integer i, i.e.,

$$-\langle F(x_k + \rho r^i d_k), d_k\rangle < \sigma \rho r^i \|d_k\|.$$

Using **Assumption A** and allowing $i \to \infty$, we get

$$-\langle F(x_{k_0}), d_{k_0}\rangle \leq 0. \tag{3.1}$$

Also from (2.7), we have

$$-\langle F(x_{k_0}), d_{k_0}\rangle \geq \|F(x_k)\|^2 > 0,$$

which contradicts (3.1). The proof is complete. ∎

The following theorem establishes the global convergence of **Algorithm 1**.

Theorem 3.2. *Let F be continuous and monotone, then the sequence $\{x_k\}$ generated by **Algorithm 1** converges globally to a solution of (1.1).*

Proof. We start by showing that the sequences $\{x_k\}$ and $\{y_k\}$ are bounded. Let x_* be an arbitrary solution of (1.1), then by monotonicity of F, we get

$$\langle F(y_k), x_k - x_*\rangle \geq \langle F(y_k), x_k - y_k\rangle. \tag{3.2}$$

Also by definition of y_k and the line search (2.6), we have

$$\langle F(y_k), x_k - y_k\rangle \geq \sigma \alpha_k \|d_k\|^2 \geq 0. \tag{3.3}$$

So, we have

$$\begin{aligned}
\|x_{k+1} - x_*\|^2 = \|P_\Omega[x_k - \zeta_k F(y_k)] - x_*\|^2 &\leq \|x_k - \zeta_k F(y_k) - x_*\| \\
&= \|x_k - x_*\|^2 - 2\zeta\langle F(y_k), x_k - x_*\rangle + t\|\zeta F(y_k)\|^2 \\
&\leq \|x_k - x_*\|^2 - 2\zeta\langle F(y_k), x_k - y_k\rangle + t\|\zeta F(y_k)\|^2 \\
&= \|x_k - x_*\|^2 - \frac{\langle F(y_k), x_k - y_k\rangle^2}{\|F(y_k)\|^2} \\
&= \|x_k - x_*\|^2 - \frac{\sigma^2 \alpha_k^4 \|d_k\|^4}{\|F(y_k)\|^4}.
\end{aligned}$$

Thus the sequence $\{\|x_k - x_*\|\}$ is non increasing and convergent, and hence $\{x_k\}$ is bounded. On the other hand (2.8) implies $\{d_k\}$ is bounded. Then, by $y_k = x_k + \alpha_k d_k$, the sequence $\{y_k\}$

is also bounded. Now, since F is continuous, there exists $M > 0$ such that $\|F(y_k)\| \leq M$ for all k. So,

$$\|x_{k+1} - x_*\|^2 \leq \|x_k - x_*\|^2 - \frac{\sigma^2 \alpha_k^4 \|d_k\|^4}{M^4}, \tag{3.4}$$

and we can deduce that

$$\lim_{k\to\infty} \alpha_k \|d_k\| = 0. \tag{3.5}$$

If $\liminf_{k\to\infty} \|d_k\| = 0$, we have $\liminf_{k\to\infty} \|F(x_k)\| = 0$. By continuity of F, the sequence $\{x_k\}$ has some accumulation point $\bar{x}$ such that $F(\bar{x}) = 0$. Since $\{\|x_k - x_*\|\}$ converges and $\bar{x}$ is an accumulation point of $\{x_k\}$, it follows that $\{x_k\}$ converges to $\bar{x}$.

If $\liminf_{k\to\infty} \|d_k\| > 0$, we have $\liminf_{k\to\infty} \|F(x_k)\| > 0$. By (3.5), it holds that $\lim_{k\to\infty} \alpha_k = 0$. Using the line search (2.6), $-F(x_k + \rho r^{m_i - 1} d_k)^T d_k < \sigma \rho r^{m_i-1} \|d_k\|^2$ and the boundedness of $\{x_k\}$, $\{d_k\}$, we can choose a subsequence such that allowing k to go to infinity in the above inequality results

$$-\langle F(\bar{x}), \bar{d} \rangle \leq 0. \tag{3.6}$$

On the other hand, from (2.7) we have

$$-\langle F(\bar{x}), \bar{d} \rangle = \|F(\bar{x})\|^2 > 0. \tag{3.7}$$

Clearly, (3.6) contradicts (3.7). Therefore, $\liminf_{k\to\infty} \|F(x_k)\| > 0$ does not hold and the proof is complete. ∎

4. Numerical Experiment

In this section, for convenience sake, we denote **Algorithm 1** by FCG method. We also divided this section into two. First we compare FCG method with CGD method [30] by solving some monotone nonlinear equations with convex constraints using different initial points and several dimensions. Secondly, the FCG method is applied to solve the ℓ_1−regularization problem that arises from compressive sensing. All codes were written in MATLAB R2017a and run on a PC with intel COREi5 processor with 4GB of RAM and CPU 2.3GHZ.

4.1. Experiment on some convex constrained nonlinear monotone equations

FCG and CGD methods have same line search implementation. The specific parameters for each method are as follows:

FCG method: $\rho = 1$, $r = 0.5$, $\sigma = 0.01$, $t = 1$ and $\beta_k = \frac{\|F(x_k)\|}{\|d_{k-1}\|}$.

CGD method: $\rho = 1$, $r = 0.39$, $\sigma = 0.0001$.

All runs were stopped whenever

$\|F(x_k)\| < 10^{-5}$.

We test problems 1 to 6 with dimensions of $n = 1000, 5000, 10,000, 50,000, 100,000$ and different initial points: $x_1 = (1, 1, ..., 1)^T$, $x_2 = (2, 2, ..., 2)^T$, $x_3 = (3, 3, ..., 3)^T$, $x_4 = (5, 5, ..., 5)^T$, $x_5 = (8, 8, ..., 8)^T$, $x_6 = (0.5, 0.5, ...0.5)^T$, $x_7 = (0.1, 0.1, ..., 0.1)^T$, $x_8 = (10, 10, ..., 10)^T$. The results of experiment reported in Tables 1-6, which contain the number of iterations (ITER), number of function evaluations (FVAL), CPU time in seconds (TIME) and the norm at the approximate solution (NORM). The symbol '−' is used to indicate that the number of iterations exceeds 1000 and/or the number of function evaluations exceeds 2000.

The problems $F(x) = (f_1(x), f_2(x), ..., f_n(x))^T$, where $x = (x_1, x_2, ..., x_n)^T$, tested are listed as follows:

Problem 1 Modified exponential function

$$\begin{aligned} F_1(x) &= e^{x_1} - 1 \\ F_i(x) &= e^{x_i} + x_{i-1} - 1 \text{ for } i = 2, 3, ..., n \\ \text{and} \quad \Omega &= \mathbb{R}^n_+. \end{aligned}$$

Problem 2 Logarithmic Function

$$F_i(x) = \ln(|x_i| + 1) - \frac{x_i}{n}, \text{ for } i = 2, 3, ..., n \quad \text{and} \quad \Omega = \mathbb{R}^n_+.$$

Problem 3 [37]

$$F_i(x) = 2x_i - \sin|x_i|, \ i = 1, 2, 3, ..., n \quad \text{and} \quad \Omega = \mathbb{R}^n_+.$$

Problem 4 Strictly convex function [26]

$$F_i(x) = e^{x_i} - 1, \text{ for } i = 2, 3, ..., n \quad \text{and} \quad \Omega = \mathbb{R}^n_+.$$

Problem 5 Linear monotone problem

$$\begin{aligned} F_1(x) &= 2.5x_1 + x_2 - 1 \\ F_i(x) &= x_{i-1} + 2.5x_i + x_{i+1} - 1 \text{ for } i = 2, 3, ..., n-1 \\ F_n(x) &= x_{n-1} + 2.5x_n - 1 \\ \text{and} \quad \Omega &= \mathbb{R}^n_+. \end{aligned}$$

Problem 6 Tridiagonal Exponential Problem [3]

$$\begin{aligned} F_1(x) &= x_1 - e^{\cos(h(x_1+x_2))} \\ F_i(x) &= x_i - e^{\cos(h(x_{i-1}+x_i+x_{i+1}))} \text{ for } i = 2, 3, ..., n-1 \\ F_n(x) &= x_n - e^{\cos(h(x_{n-1}+x_n))}, \\ \text{where } h &= \frac{1}{n+1} \\ \text{and} \quad \Omega &= \mathbb{R}^n_+. \end{aligned}$$

The results of the numerical performance indicate that the FCG method is more efficient than the CGD method for the given test problems as it solves more problems than CGD method which fails to solve most of the problems. In particular CGD method fails to solve problems 5 and 6 completely while FCG was able to solve the problems. Thus, FCG method is an effective tool for solving nonlinear monotone equations with convex constraints, especially for large-scale problems.

Table 1. Numerical Results for FCG and CGD for Problem 1 with given initial points and dimensions

DIMENSION	INITIAL POINT	FCG				CGD			
		ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	x_1	30	151	0.029034	9.97E-06	-	-	-	-
	x_2	32	162	0.032225	8E-06	-	-	-	-
	x_3	29	147	0.024743	8.25E-06	-	-	-	-
	x_4	30	154	0.024078	8.98E-06	-	-	-	-
	x_5	32	169	0.024838	9.5E-06	-	-	-	-
	x_6	29	146	0.020169	9.09E-06	3	21	0.002236	0
	x_7	26	131	0.019619	6.87E-06	3	21	0.001834	0
	x_8	117	519	0.058816	9.57E-06	3	21	0.001666	0
5000	x_1	27	136	0.07062	9.27E-06	-	-	-	-
	x_2	29	147	0.069378	6.35E-06	-	-	-	-
	x_3	26	132	0.062323	9.74E-06	-	-	-	-
	x_4	27	139	0.056432	7.86E-06	-	-	-	-
	x_5	29	154	0.073783	9.28E-06	-	-	-	-
	x_6	26	131	0.057241	9.8E-06	3	21	0.00279	0
	x_7	24	121	0.061208	6.53E-06	3	21	0.001789	0
	x_8	225	945	0.359843	6.08E-06	3	21	0.001659	0
10000	x_1	27	136	0.110866	6.55E-06	-	-	-	-
	x_2	28	142	0.105131	6.99E-06	-	-	-	-
	x_3	26	132	0.095317	7.65E-06	-	-	-	-
	x_4	26	134	0.096757	9.22E-06	-	-	-	-
	x_5	29	154	0.10975	6.59E-06	-	-	-	-
	x_6	26	131	0.091813	7.43E-06	3	21	0.002557	0
	x_7	23	116	0.081306	9.74E-06	3	21	0.001912	0
	x_8	207	872	0.667213	9.86E-06	3	21	0.002028	0
50000	x_1	27	136	0.416897	5.45E-06	-	-	-	-
	x_2	28	142	0.4511	5.54E-06	-	-	-	-
	x_3	26	132	0.41667	7.7E-06	-	-	-	-
	x_4	26	134	0.42627	8.15E-06	-	-	-	-
	x_5	29	154	0.504924	5.43E-06	-	-	-	-
	x_6	26	131	0.414027	7.1E-06	3	21	0.002425	0
	x_7	24	121	0.44835	6.42E-06	3	21	0.001936	0
	x_8	193	816	2.51444	9.92E-06	3	21	0.001939	0
100000	x_1	27	136	0.991751	6.37E-06	-	-	-	-
	x_2	28	142	1.260811	6.4E-06	-	-	-	-
	x_3	26	132	1.424801	9.41E-06	-	-	-	-
	x_4	26	134	1.530526	9.67E-06	-	-	-	-
	x_5	29	154	1.381487	6.31E-06	-	-	-	-
	x_6	26	131	1.004984	8.59E-06	3	21	0.002641	0
	x_7	24	121	0.854936	8.11E-06	3	21	0.003401	0
	x_8	78	370	2.377291	9.74E-06	3	21	0.001946	0

Table 2. Numerical Results for FCG and CGD for Problem 2 with given initial points and dimensions

DIMENSION	INITIAL POINT	FCG				CGD			
		ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	x_1	6	19	0.004914	3.6E-08	3	10	0.004752	0
	x_2	7	22	0.006036	1.74E-08	-	-	-	-
	x_3	7	22	0.004458	2.21E-06	-	-	-	-
	x_4	8	25	0.004995	5.45E-06	-	-	-	-
	x_5	10	31	0.009144	8.47E-08	-	-	-	-
	x_6	5	16	0.004309	4.37E-07	12	37	0.003022	0
	x_7	4	13	0.003884	5.17E-07	10	31	0.003156	0
	x_8	11	34	0.008439	2.64E-08	10	31	0.002395	0
5000	x_1	6	19	0.013848	6.26E-09	3	10	0.010011	0
	x_2	7	22	0.015027	2.36E-09	-	-	-	-
	x_3	7	22	0.01877	8.93E-07	-	-	-	-
	x_4	8	25	0.01957	2.58E-06	-	-	-	-
	x_5	10	31	0.023509	1.74E-08	6	19	0.018264	0
	x_6	5	16	0.011425	1.42E-07	12	37	0.003034	0
	x_7	4	13	0.008892	1.75E-07	10	31	0.002234	0
	x_8	11	34	0.019901	3.7E-09	10	31	0.002247	0
10000	x_1	6	20	0.025059	3.62E-09	3	10	0.017026	0
	x_2	7	23	0.027807	1.24E-09	-	-	-	-
	x_3	7	22	0.0271	6.86E-07	-	-	-	-
	x_4	8	25	0.024772	2.22E-06	-	-	-	-
	x_5	10	32	0.034177	1.07E-08	12	48	0.056075	0
	x_6	5	17	0.021084	9.73E-08	12	37	0.004599	0
	x_7	4	13	0.016566	1.21E-07	10	31	0.002371	0
	x_8	11	35	0.038309	2E-09	10	31	0.002892	0
50000	x_1	8	29	0.113023	8.3E-06	3	10	0.066789	0
	x_2	7	24	0.092315	1E-05	-	-	-	-
	x_3	17	64	0.238282	5.77E-06	-	-	-	-
	x_4	19	71	0.27681	7.15E-06	-	-	-	-
	x_5	14	49	0.198167	6.54E-06	-	-	-	-
	x_6	12	46	0.167172	7.79E-06	12	37	0.003578	0
	x_7	11	43	0.154196	9.67E-06	10	31	0.002085	0
	x_8	12	40	0.161888	8.52E-06	10	31	0.002224	0
100000	x_1	9	33	0.237879	5.71E-06	3	10	0.123028	0
	x_2	8	28	0.211838	6.74E-06	-	-	-	-
	x_3	17	64	0.507595	8.14E-06	-	-	-	-
	x_4	20	75	0.580962	5.05E-06	-	-	-	-
	x_5	14	49	0.394332	9.09E-06	-	-	-	-
	x_6	13	50	0.380856	5.48E-06	12	37	0.003665	0
	x_7	12	47	0.348441	6.8E-06	10	31	0.002801	0
	x_8	13	44	0.392105	5.8E-06	10	31	0.002108	0

Table 3. Numerical Results for FCG and CGD for Problem 3 with given initial points and dimensions

DIMENSION	INITIAL POINT	**FCG**				**CGD**			
		ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
	x_1	23	93	0.010923	6.1E-06	13	42	0.010427	0
	x_2	23	93	0.013051	6.55E-06	-	-	-	-
	x_3	20	81	0.01293	8.5E-06	9	39	0.009423	0
	x_4	24	98	0.014461	5.63E-06	-	-	-	-
	x_5	23	93	0.01694	7.07E-06	-	-	-	-
1000	x_6	22	89	0.021464	7.14E-06	7	26	0.001876	0
	x_7	20	81	0.014054	6.02E-06	10	42	0.002722	0
	x_8	25	103	0.017473	5.94E-06	9	39	0.002248	0
	x_1	24	97	0.041347	6.82E-06	-	-	-	-
	x_2	24	97	0.040248	7.32E-06	-	-	-	-
	x_3	21	85	0.037918	9.51E-06	-	-	-	-
	x_4	25	102	0.043526	6.29E-06	-	-	-	-
	x_5	24	97	0.04303	7.9E-06	-	-	-	-
5000	x_6	23	93	0.03983	7.98E-06	7	26	0.002126	0
	x_7	21	85	0.038976	6.73E-06	10	42	0.003248	0
	x_8	26	107	0.044616	6.64E-06	9	39	0.002423	0
	x_1	24	97	0.069348	9.65E-06	-	-	-	-
	x_2	25	101	0.077087	5.18E-06	-	-	-	-
	x_3	22	89	0.065376	6.72E-06	-	-	-	-
	x_4	25	102	0.079566	8.9E-06	-	-	-	-
	x_5	25	101	0.072858	5.59E-06	-	-	-	-
10000	x_6	24	97	0.070883	5.64E-06	7	26	0.002492	0
	x_7	21	85	0.058127	9.52E-06	10	42	0.00411	0
	x_8	26	107	0.086316	9.39E-06	9	39	0.002452	0
	x_1	26	105	0.326713	5.39E-06	-	-	-	-
	x_2	26	105	0.300343	5.79E-06	-	-	-	-
	x_3	23	93	0.271562	7.52E-06	-	-	-	-
	x_4	26	106	0.308558	9.95E-06	-	-	-	-
	x_5	26	105	0.342744	6.25E-06	-	-	-	-
50000	x_6	25	101	0.310528	6.31E-06	7	26	0.002338	0
	x_7	23	93	0.266319	5.32E-06	10	42	0.002626	0
	x_8	28	115	0.3389	5.25E-06	9	39	0.00371	0
	x_1	26	105	0.609266	7.63E-06	-	-	-	-
	x_2	26	105	0.640604	8.19E-06	-	-	-	-
	x_3	24	97	0.604267	5.31E-06	-	-	-	-
	x_4	27	110	0.666098	7.04E-06	-	-	-	-
	x_5	26	105	0.622149	8.84E-06	-	-	-	-
100000	x_6	25	101	0.621465	8.92E-06	7	26	0.002508	0
	x_7	23	93	0.567894	7.52E-06	10	42	0.00269	0
	x_8	28	115	0.724637	7.42E-06	9	39	0.002607	0

Table 4. Numerical Results for FCG and CGD for Problem 4 with given initial points and dimensions

DIMENSION	INITIAL POINT	FCG				CGD			
		ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	x_1	21	85	0.010375	7.37E-06	-	-	-	-
	x_2	22	90	0.01171	7.98E-06	11	45	0.008779	0
	x_3	22	91	0.009893	9.46E-06	3	21	0.004732	0
	x_4	22	93	0.016824	7.88E-06	3	21	0.004487	0
	x_5	23	102	0.012653	5.28E-06	3	21	0.005234	0
	x_6	21	85	0.016972	8.87E-06	3	21	0.001853	0
	x_7	20	81	0.012323	5.45E-06	3	21	0.00177	0
	x_8	2	18	0.005313	0	3	21	0.001968	0
5000	x_1	22	89	0.037557	8.24E-06	-	-	-	-
	x_2	23	94	0.033656	8.93E-06	-	-	-	-
	x_3	24	99	0.033578	5.29E-06	3	21	0.010306	0
	x_4	23	97	0.035841	8.81E-06	3	21	0.012105	0
	x_5	24	106	0.047643	5.9E-06	3	21	0.012359	0
	x_6	22	89	0.03202	9.91E-06	3	21	0.00166	0
	x_7	21	85	0.050125	6.1E-06	3	21	0.001721	0
	x_8	2	18	0.009803	0	3	21	0.001635	0
10000	x_1	23	93	0.065522	5.83E-06	-	-	-	-
	x_2	24	98	0.053947	6.31E-06	-	-	-	-
	x_3	24	99	0.060434	7.48E-06	3	21	0.018915	0
	x_4	24	101	0.05745	6.23E-06	3	21	0.017853	0
	x_5	24	106	0.074314	8.35E-06	3	21	0.016733	0
	x_6	23	93	0.054277	7.01E-06	3	21	0.001641	0
	x_7	21	85	0.057827	8.62E-06	3	21	0.00238	0
	x_8	2	18	0.018646	0	3	21	0.001563	0
50000	x_1	24	97	0.264969	6.51E-06	-	-	-	-
	x_2	25	102	0.242806	7.06E-06	-	-	-	-
	x_3	25	103	0.253747	8.36E-06	3	21	0.072756	0
	x_4	25	105	0.242377	6.96E-06	3	21	0.071677	0
	x_5	25	110	0.24953	9.33E-06	3	21	0.072242	0
	x_6	24	97	0.235073	7.84E-06	3	21	0.001643	0
	x_7	22	89	0.214897	9.64E-06	3	21	0.002218	0
	x_8	2	18	0.064875	0	3	21	0.002332	0
100000	x_1	24	97	0.450026	9.21E-06	-	-	-	-
	x_2	25	102	0.496804	9.98E-06	-	-	-	-
	x_3	26	107	0.529191	5.91E-06	3	21	0.155644	0
	x_4	25	105	0.515341	9.85E-06	3	21	0.146987	0
	x_5	26	114	0.553523	6.6E-06	3	21	0.141577	0
	x_6	25	101	0.465119	5.54E-06	3	21	0.001987	0
	x_7	23	93	0.420296	6.82E-06	3	21	0.001702	0
	x_8	2	18	0.135172	0	3	21	0.002578	0

Table 5. Numerical Results for FCG and CGD for Problem 5 with given initial points and dimensions

DIMENSION	INITIAL POINT	FCG				CGD			
		ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	x_1	152	802	0.075299	9.2E-06	-	-	-	-
	x_2	139	736	0.073897	7.3E-06	-	-	-	-
	x_3	136	726	0.072044	6.7E-06	-	-	-	-
	x_4	165	872	0.093292	7.4E-06	-	-	-	-
	x_5	165	873	0.084202	8.2E-06	-	-	-	-
	x_6	150	793	0.080311	7.2E-06	-	-	-	-
	x_7	153	808	0.077297	7.4E-06	-	-	-	-
	x_8	183	964	0.09986	7E-06	-	-	-	-
5000	x_1	151	796	0.302958	9.1E-06	-	-	-	-
	x_2	108	580	0.222843	8.5E-06	-	-	-	-
	x_3	133	711	0.244279	9.8E-06	-	-	-	-
	x_4	155	822	0.268972	9.7E-06	-	-	-	-
	x_5	177	934	0.331852	7.1E-06	-	-	-	-
	x_6	143	758	0.247833	1E-05	-	-	-	-
	x_7	145	768	0.260931	9.8E-06	-	-	-	-
	x_8	177	934	0.376048	9.4E-06	-	-	-	-
10000	x_1	151	796	0.582134	9.1E-06	-	-	-	-
	x_2	101	545	0.405084	7.5E-06	-	-	-	-
	x_3	150	796	0.59398	9.5E-06	-	-	-	-
	x_4	172	908	0.666833	9.1E-06	-	-	-	-
	x_5	158	839	0.60491	9.3E-06	-	-	-	-
	x_6	159	839	0.594092	9.9E-06	-	-	-	-
	x_7	150	793	0.561624	9.7E-06	-	-	-	-
	x_8	171	905	0.651769	7.7E-06	-	-	-	-
50000	x_1	147	776	2.228341	9E-06	-	-	-	-
	x_2	89	486	1.463798	8.6E-06	-	-	-	-
	x_3	147	780	2.378053	9.1E-06	-	-	-	-
	x_4	156	829	2.491981	6.7E-06	-	-	-	-
	x_5	176	930	2.74415	8.2E-06	-	-	-	-
	x_6	177	930	2.722724	8.6E-06	-	-	-	-
	x_7	151	799	2.321288	7.7E-06	-	-	-	-
	x_8	159	845	2.448193	9.8E-06	-	-	-	-
100000	x_1	148	782	5.294027	7.9E-06	-	-	-	-
	x_2	89	486	3.293977	7.9E-06	-	-	-	-
	x_3	132	707	4.714172	9.5E-06	-	-	-	-
	x_4	144	769	5.166183	9.4E-06	-	-	-	-
	x_5	165	875	5.898258	7.3E-06	-	-	-	-
	x_6	171	900	5.94172	8E-06	-	-	-	-
	x_7	152	805	5.387028	6.7E-06	-	-	-	-
	x_8	165	875	5.867254	9.7E-06	-	-	-	-

Table 6. Numerical Results for FCG and CGD for Problem 6 with given initial points and dimensions

DIMENSION	INITIAL POINT	FCG				CGD			
		ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	x_1	24	97	0.018177	6.47E-06	-	-	-	-
	x_2	23	93	0.020801	5.41E-06	-	-	-	-
	x_3	21	85	0.018842	8.49E-06	-	-	-	-
	x_4	24	97	0.017617	8.59E-06	-	-	-	-
	x_5	25	101	0.018456	9.94E-06	-	-	-	-
	x_6	24	97	0.017399	8.35E-06	-	-	-	-
	x_7	24	97	0.018177	9.86E-06	-	-	-	-
	x_8	26	105	0.028836	6.85E-06	-	-	-	-
5000	x_1	25	101	0.071229	7.24E-06	-	-	-	-
	x_2	24	97	0.069072	6.05E-06	-	-	-	-
	x_3	22	89	0.062428	9.5E-06	-	-	-	-
	x_4	25	101	0.093955	9.62E-06	-	-	-	-
	x_5	27	109	0.072351	5.56E-06	-	-	-	-
	x_6	25	101	0.065859	9.35E-06	-	-	-	-
	x_7	26	105	0.090732	5.52E-06	-	-	-	-
	x_8	27	109	0.071772	7.67E-06	-	-	-	-
10000	x_1	26	105	0.154804	5.12E-06	-	-	-	-
	x_2	24	97	0.117887	8.56E-06	-	-	-	-
	x_3	23	93	0.114087	6.72E-06	-	-	-	-
	x_4	26	105	0.148228	6.8E-06	-	-	-	-
	x_5	27	109	0.144108	7.87E-06	-	-	-	-
	x_6	26	105	0.129431	6.61E-06	-	-	-	-
	x_7	26	105	0.130249	7.8E-06	-	-	-	-
	x_8	28	113	0.149137	5.43E-06	-	-	-	-
50000	x_1	27	109	0.635653	5.73E-06	-	-	-	-
	x_2	25	101	0.533873	9.57E-06	-	-	-	-
	x_3	24	97	0.550562	7.51E-06	-	-	-	-
	x_4	27	109	0.605892	7.6E-06	-	-	-	-
	x_5	28	113	0.636995	8.8E-06	-	-	-	-
	x_6	27	109	0.613244	7.39E-06	-	-	-	-
	x_7	27	109	0.602482	8.72E-06	-	-	-	-
	x_8	29	117	0.645397	6.07E-06	-	-	-	-
100000	x_1	27	109	1.250775	8.1E-06	-	-	-	-
	x_2	26	105	1.246817	6.77E-06	-	-	-	-
	x_3	25	101	1.158025	5.31E-06	-	-	-	-
	x_4	28	113	1.301028	5.38E-06	-	-	-	-
	x_5	29	117	1.338954	6.22E-06	-	-	-	-
	x_6	28	113	1.324453	5.23E-06	-	-	-	-
	x_7	28	113	1.299459	6.17E-06	-	-	-	-
	x_8	29	117	1.355021	8.58E-06	-	-	-	-

4.2. Experiments on the ℓ_1-norm regularization problem in compressive sensing

There are many problems in signal processing and statistical inference involving finding sparse solutions to ill-conditioned linear systems of equations. Among popular approach is minimizing an objective function which contains quadratic (ℓ_2) error term and a sparse ℓ_1-regularization term, i.e.,

$$\min_x \frac{1}{2}\|y - Ax\|_2^2 + \omega\|x\|_1, \tag{4.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^k$ is an observation, $A \in \mathbb{R}^{k\times n}$ ($k << n$) is a linear operator, ω is a nonnegative parameter, $\|x\|_2$ denotes the Euclidean norm of x and $\|x\|_1 = \sum_{i=1}^n |x_i|$ is the ℓ_1-norm of x. It is easy to see that problem (4.1) is a convex unconstrained minimization problem. Due to the fact that if the original signal is sparse or approximately sparse in some orthogonal basis, problem (4.1) frequently appears in compressive sensing, and hence an exact restoration can be produced by solving (4.1).

Iterative methods for solving (4.1) have been been presented in many literatures, (see [11, 13, 2, 12, 25, 4]). The most popular method among these methods is the gradient based method and the earliest gradient projection method for sparse reconstruction (GPRS) was proposed by Figueiredo et al. [12]. The first step of the GPRS method is to express (4.1) as a quadratic problem using the following process. Let $x \in \mathbb{R}^n$ and splitting it into its positive and negative parts. Then x can be formulated as

$$x = u - v, \qquad u \geq 0, \quad v \geq 0,$$

where $u_i = (x_i)_+$, $v_i = (-x_i)_+$ for all $i = 1, 2, ..., n$, and $(.)_+ = \max\{0, .\}$. By definition of ℓ_1-norm, we have $\|x\|_1 = e_n^T u + e_n^T v$, where $e_n = (1, 1, ..., 1)^T \in \mathbb{R}^n$. Now (4.1) can be written as

$$\min_{u,v} \frac{1}{2}\|y - A(u - v)\|_2^2 + \omega e_n^T u + \omega e_n^T v, \qquad u \geq 0, \quad v \geq 0, \tag{4.2}$$

which is a bound-constrained quadratic program. However, from [12], equation (4.2) can be written in standard form as

$$\min_z \frac{1}{2} z^T Bz + c^T z, \qquad \text{such that} \quad z \geq 0, \tag{4.3}$$

where $z = \begin{pmatrix} u \\ v \end{pmatrix}$, $\quad c = \omega e_{2n} + \begin{pmatrix} -b \\ b \end{pmatrix}$, $\quad b = A^T y$, $\quad B = \begin{pmatrix} A^T A & -A^T A \\ -A^T A & A^T A \end{pmatrix}$.
Clearly, B is a positive semidefinite matrix, which implies that equation (4.3) is a convex quadratic problem.

Xiao et al. [30] translated (4.3) into a linear variable inequality problem which is equivalent to a linear complementarity problem. Furthermore, they pointed out that z is a solution of the linear complementarity problem if and only if it is a solution of the nonlinear equation:

$$F(z) = \min\{z, Bz + c\} = 0. \tag{4.4}$$

It was proved in [29, 23] that $F(z)$ is continuous and monotone. Therefore problem (4.1) can be translated into problem (1.1) and thus FCG method can be applied to solve (4.1).

In this experiment, we consider a simple compressive sensing possible situation, where our goal is to reconstruct a sparse signal of length n from k observations. The quality of restoration is assessed by mean of squared error (MSE) to the original signal $\bar{x}$,

$$MSE = \frac{1}{n}\|\bar{x} - x_*\|^2,$$

where x_* is the recovered or restored signal. The signal size is choosen as $n = 2^{12}$, $k = 2^{10}$ and the original signal contains 2^7 randomly nonzero elements. A is the Gaussian matrix generated by the command $rand(m, n)$ in MATLAB. In addition, the measurement y is distributed with noise, that is, $y = A\not{x} + \mu$, where μ is the Gaussian noise distributed normally with mean 0 and variance 10^{-4} ($N(0, 10^{-4})$).

To show the performance of the FCG method in compressive sensing, we compare it with the CGD method. The parameters in both FCG and CGD methods are chosen as $\rho = 10$, $\sigma = 10^{-4}$ and $r = 0.5$, which came from [30]. After series of experiments, we observe that for FCG method, the parameter η has a great impact on the restoration of signal. Finally, we choose $\eta = 0.2$ in our experiment and the merit function used is $f(x) = \frac{1}{2}\|y - Ax\|_2^2 + \omega\|x\|_1$. To achieve fairness in comparison, each code was run from same initial point, same continuation technique on the parameter ω, and observed only the behaviour of the convergence of each method to have a similar accurate solution. The experiment is initialized by $x_0 = A^T y$ and terminates when

$$\frac{\|f_k - f_{k-1}\|}{\|f_{k-1}\|} < 10^{-5},$$

where f_k is the function evaluation at x_k.

In Fig. 1, FCG and CGD methods recovered the disturbed signal almost exactly. In order to show visually the performance of both methods, four figures were plotted to demonstrate their convergence behaviour based on MSE, objective function values, number of iterations and CPU time, see Fig. $2-5$. Furthermore, the experiment was repeated for 10 different noise samples and the average was also computed, see Table 7. From the Table, it can be observed that the FCG is more efficient as it has fewer iterations and CPU time than CGD method in most cases.

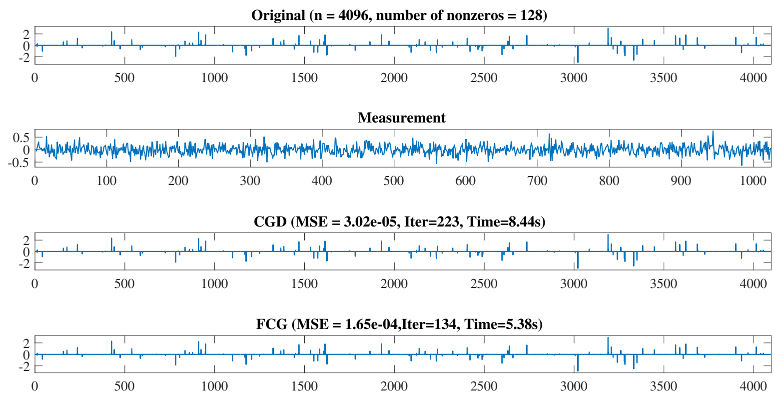

Fig. 1. From top to bottom: the original image, the measurement, and the recovered signals by CGD and FCG methods.

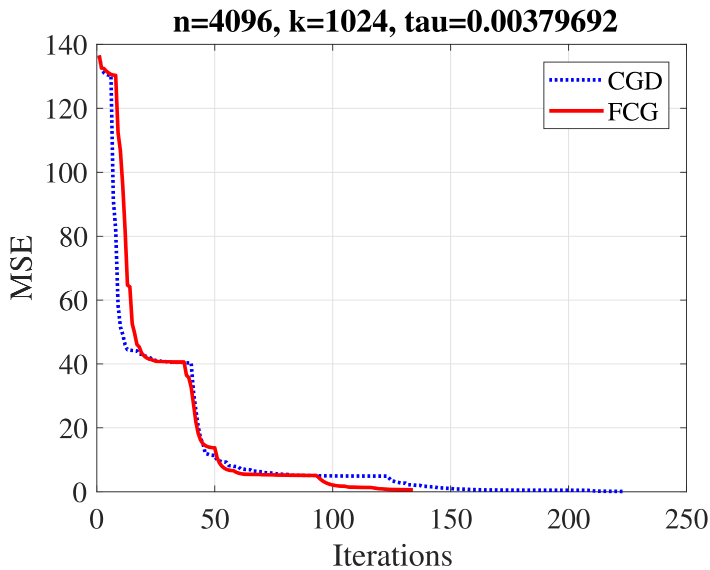

Fig. 2. Iterations

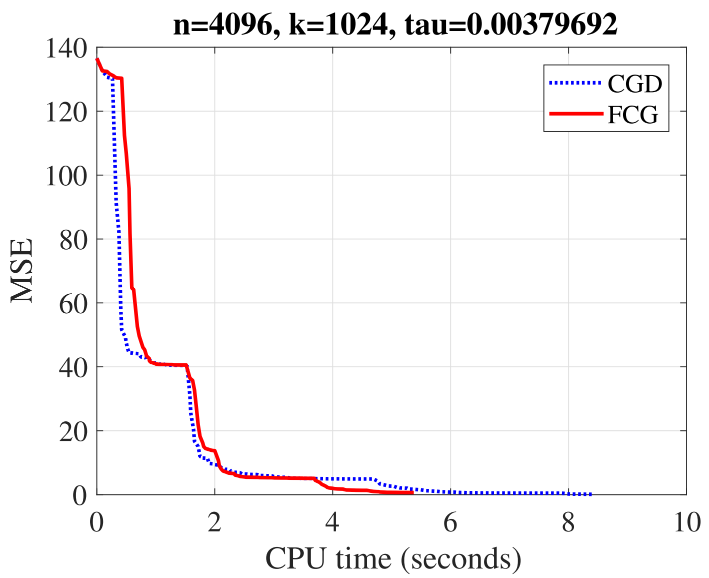

Fig. 3. CPU time (seconds)

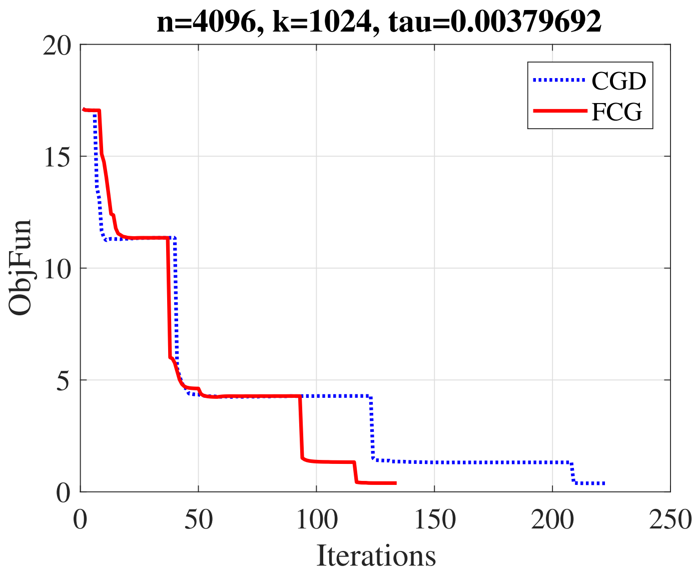

Fig. 4. Iterations

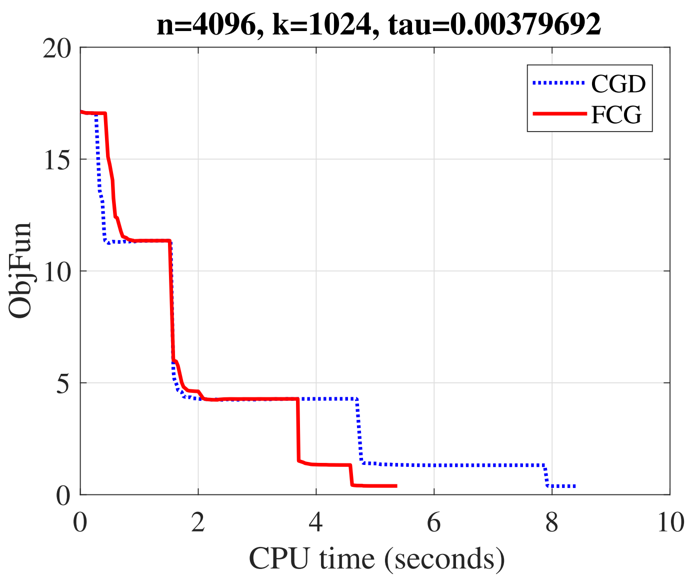

Fig. 5. CPU time (seconds)

Table 7. Ten experiment results together with average result of ℓ_1-norm regularization problem for FCG and CGD methods

	FCG			CGD		
	MSE	ITER	CPU(s)	MSE	ITER	CPU(s)
	2.31E-04	100	3.98	3.40E-05	196	7.31
	1.65E-04	134	5.38	3.02E-05	223	8.44
	1.40E-04	130	5.14	5.21E-05	164	6.3
	1.65E-04	134	5.59	3.02E-05	223	8.69
	1.75E-04	127	4.83	4.48E-05	218	8.14
$\eta = 0.2$	6.78E-04	169	6.38	1.85E-05	215	8.44
	1.47E-04	137	5.28	4.94E-05	191	8.66
	2.72E-04	94	4.53	4.33E-05	224	8.83
	1.67E-04	117	4.89	1.26E-05	135	5.55
	1.07E-04	119	4.64	2.78E-05	181	6.91
Average	2.25E-04	126.1	5.064	3.43E-05	197	7.727

5. Conclusions

In this article, a family of conjugate gradient projection method for solving nonlinear monotone equation with convex constraints was proposed. The proposed method is suitable for for solving nonsmooth equations as it does not require Jacobian information of the nonlinear equations. The global convergence of the proposed method was established under suitable conditions.

We can view the the proposed method as an extension of the method in [10] to solve convex constrained problems. Numerical results show that the proposed method is more efficient than the CGD method for the given constrained problems. Furthermore, the proposed method can be applied to solve ℓ_1-norm regularization problem in compressive sensing.

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Idempotent and Regular Elements on Some Semigroups of the Generalized Cohypersubstitutions of type $\tau = (2)$

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ABSTRACT

A generalized cohypersubstitution σ of type $\tau = (n_i)_{i \in I}$ is a mapping which maps every n_i-ary cooperation symbol f_i to the coterm $\sigma(f)$ of type τ. We denoted the set of all generalized cohypersubstitutions of type τ by $Cohyp_G(\tau)$. In this study, we focus on the semigroups $(Cohyp_G(2), +_{CG})$ and $(Cohyp_G(2), \oplus_{CG})$ where $+_{CG}$ and $\oplus_{CG}$ are binary operations the set $Cohyp_G(2)$. We characterize the set of all idempotent and regular elements of these semigroups.

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1. Introduction

The concept of cohypersubstitution of type τ was first introduced by K. Denecke and K. Saengsura [3] in 2009. They used as the main tool in the study of cohyperidentities. They defined coalgebras, coidentities, cohyperidentities and applied all the concepts to construct the monoid of cohypersubstitutions of type τ. After that, in 2013, S. Jermjitpornchai and N. Seangsura [5] generalized the concepts of K. Denecke and K. Saengsura [3] by studying on the generalized cohypersubstitutions of type $\tau = (n_i)_{i \in I}$, introduced coterms, generalized superpositions, some algebraic-structural properties and constructed the monoid of generalized cohypersubstitutions. Later that, in the same year, N. Seangsura and S. Jermjitpornchai [8] fixed type $\tau = (2)$ and characterized all idempotent and regular elements of the generalized cohypersubstitutions of type $\tau = (2)$. After the study, the structural properties and special elements of the monoid of generalized cohypersubstitutions of type $\tau = (2), \tau = (3)$ and $\tau = (n)$ have been stydied by many other authors, see in [1], [5] and [8]. Moreover, in 2021, N. Chansuriya and S. Phuapong gave some structural properties and the relationship among submonoids of the monoid of generalized cohypersubstitutions of type τ by using the concepts in [6] and [7]. They also defined two new binary operations $+_{CG}$ and $\oplus_{CG}$ on the set of all generalized cohypersubstitutions of type τ, $Cohyp_G(\tau)$, and showed that

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$(Cohyp_G(\tau), +_{CG})$,$(Cohyp_G(\tau), \oplus_{CG})$ were semigroups.

In this study, we fix type $\tau = (2)$ and focus on the semigroups $(Cohyp_G(2), +_{CG})$ and $(Cohyp_G(2), \oplus_{CG})$. We characterize the set of all idempotent and regular elements of this semigroups.

2. Preliminaries

In this section, we provide the basic concept of the monoid of set of all generalized cohypersubstitutions which is very useful to this research.

Let A be a non-empty set and $n \in \mathbb{N}^+ = \mathbb{N} \cup \{0\}$. Define the union of n disjoint copies of A by $A^{\sqcup n} := \underline{n} \times A$ where $\underline{n} = \{1, 2, \ldots, n\}$, so it is called the n-th copower of A. An element (i, a) in this copower corresponds to the element a in the i-th copy of A where $i \in \underline{n}$. A mapping $f^A : A \to A^{\sqcup n}$ is a co-operation on A; the natural number n is called the arity of the co-operation f^A. Every n-ary co-operation f^A on the set A can be uniquely expressed as the pair of mappings (f_1^A, f_2^A) where $f_1^A : A \to \underline{n}$ gives the labelling used by f^A in mapping elements to copies of A, and $f_2^A : A \to A$ shows what element of A any element is mapped to, so $f^A(a) = (f_1^A(a), f_2^A(a))$. We denote the set of all n-ary co-operations defined on A by $cO_A^{(n)} = \{f^A : A \to A^{\sqcup n}\}$.

Let $\tau = (n_i)_{i \in I}$ and let $(f_i)_{i \in I}$ be an indexed set of co-operation symbols which f_i has arity n_i for each $i \in I$. Let $\bigcup \{e_j^n \mid n \geq 1, n \in \mathbb{N}^+, 0 \leq j \leq n-1\}$ be a set of symbols which disjoint from $\{f_i \mid i \in I\}$ such that e_j^n has arity n for each $0 \leq j \leq n-1$. An *coterms* of type τ are defined as follows:

(i) For every $i \in I$, the co-operation symbol f_i is an n-ary coterm of type τ.

(ii) For every $n \geq 1$ and $0 \leq j \leq n-1$ the symbol e_j^n is an n-ary coterm of type τ.

(iii) If $t_1, \ldots, t_{n_i}$ are n-ary coterms of type τ, then $f_i[t_1, \ldots, t_{n_i}]$ is an n-ary coterm of type τ for every $i \in I$, and if $t_0, \ldots, t_{n-1}$ are m-ary coterms of type τ, then $e_j^n[t_0, \ldots, t_{n-1}]$ is an n-ary coterm of type τ for every $0 \leq j \leq n-1$.

Let $CT_\tau^{(n)}$ be the set of all n-ary coterms of type τ, and $CT_\tau := \bigcup\limits_{n \geq 1} CT_\tau^{(n)}$ the set of all coterms of type τ.

For example, let us consider the type $\tau = (2)$ with one binary co-operation symbol f and the set of all injection symbols $E := \{e_j^n \mid n, j \in \mathbb{N}^+ := \mathbb{N} \cup \{0\}\}$. Then some example of coterm of type $\tau = (2)$ are:

$$e_0^2, e_1^2, f[e_0^2, e_1^2], f[e_1^2, e_2^2], f[f[e_1^2, e_0^2], e_2^2], f[e_0^2, f[e_1^2, e_1^2]], f[f[e_0^2, e_3^2, f[e_1^2, e_4^2]].$$

Definition 2.1. [5] Let $m \in \mathbb{N}^+ = \mathbb{N} \cup \{0\}$. A generalized superposition of coterms $S^m : CT_\tau^{m+1} \to CT_\tau$ is defined inductively by the following steps:

(i) If $t = e_i^n$ and $0 \leq i \leq m-1$, then $S^m(e_i^n, t_0, \ldots, t_{m-1}) = t_i$, where $t_0, \ldots, t_{m-1} \in CT_\tau$.

(ii) If $t = e_i^n$ and $0 < m \leq i \leq n-1$, then $S^m(e_i^n, t_0, \ldots, t_{m-1}) = e_i^n$, where $t_0, \ldots, t_{m-1} \in CT_\tau$.

(iii) If $t = f_i[s_1, \dots, s_{n_i}]$, then $S^m(t, t_1, \dots, t_m) = f_i(S^m(s_1, t_1, \dots, t_m), \dots, S^m(s_{n_i}, t_1, \dots, t_m))$, where $S^m(s_1, t_1, \dots, t_m), \dots, S^m(s_{n_i}, t_1, \dots, t_m) \in CT_\tau$.

The above definition can be written as the following forms:

(i) If $t = e_i^n$ and $0 \le i \le m-1$, then $e_i^n[t_0, \dots, t_{m-1}] = t_i$, where $t_0, \dots, t_{m-1} \in CT_\tau$.

(ii) If $t = e_i^n$ and $0 < m \le i \le n-1$, then $e_i^n[t_0, \dots, t_{m-1}] = e_i^n$, where $t_0, \dots, t_{m-1} \in CT_\tau$.

(iii) If $t = f_i[s_1, \dots, s_{n_i}]$, then $(f_i[s_1, \dots, s_{n_i}])[t_1, \dots, t_m] = f_i(s_1[t_1, \dots, t_m], \dots, s_{n_i}[t_1, \dots, t_m])$, where $s_1[t_1, \dots, t_m], \dots, s_{n_i}[t_1, \dots, t_m] \in CT_\tau$.

Definition 2.2. [5] A generalized cohypersubstitution of type τ is a mapping $\sigma : \{f_i \mid i \in I\} \to CT_\tau$. The extension of σ is a mapping $\widehat{\sigma} : CT_\tau \to CT_\tau$ which is inductively defined by the following steps :

(i) $\widehat{\sigma}(e_j^n) := e_j^n$ for every $n \ge 1$ and $0 \le j \le n-1$,

(ii) $\widehat{\sigma}(f_i) := \sigma(f_i)$ for every $i \in I$,

(iii) $\widehat{\sigma}(f_i[t_1, \dots, t_{n_i}]) := \sigma(f_i)[\widehat{\sigma}(t_1), \dots, \widehat{\sigma}(t_{n_i})]$ for $t_1, \dots t_{n_i} \in CT_\tau^{(n)}$.

Let $Cohyp_G(\tau)$ be the set of all generalized cohypersubstitutions of type τ.

Proposition 2.3. [5] *If $t, t_1, \dots, t_n \in CT_\tau$ and $\sigma \in Cohyp_G(\tau)$, then*

$$\widehat{\sigma}(t[t_1, \dots, t_n]) = \widehat{\sigma}(t)[\widehat{\sigma}(t_1), \dots, \widehat{\sigma}(t_n)].$$

On the set $Cohyp_G(\tau)$ of all generalized cohypersubstitutions of type τ, we may define an operation $\circ_{CG} : Cohyp_G(\tau) \times Cohyp_G(\tau) \to Cohyp_G(\tau)$ by $\sigma_1 \circ_{CG} \sigma_2 := \widehat{\sigma_1} \circ \sigma_2$ for all $\sigma_1, \sigma_2 \in Cohyp_G(\tau)$ where $\circ$ is the usual composition of mappings. Let σ_{id} be the generalized cohypersubstitution such that $\sigma_{id}(f_i) := f_i[e_0^n, e_1^n, \dots, e_{n_i-1}^n]$ for all $i \in I$. Then σ_{id} is an identity element in $Cohyp_G(\tau)$. Thus $\underline{Cohyp_G(\tau)} := (Cohyp_G(\tau), \circ_{CG}, \sigma_{id})$ is a monoid and called the *monoid of generalized cohypersubstitutions of type τ*. A algebraic-structural properties of the monoid $\underline{Cohyp_G(\tau)}$ can be found in [5].

In [2], a new binary operation "$+_{CG}$" on the set $Cohyp_G(\tau)$ was defined by

$$(\sigma_1 +_{CG} \sigma_2)(f_i) := \sigma_2(f_i)[\underbrace{\sigma_1(f_i), \dots, \sigma_1(f_i)}_{n_i - terms}] \in CT_{(\tau)},$$

for all $\sigma_1, \sigma_2 \in Cohyp_G(\tau)$. Then $(Cohyp_G(\tau), +_{CG})$ is a semigroup. Furthermore, they also defined another new binary operation "$\oplus_{CG}$" on the set $Cohyp_G(\tau)$ by

$$(\sigma_1 \oplus_{CG} \sigma_2)(f_i) := \sigma_1(f_i)[\underbrace{\sigma_2(f_i), \dots, \sigma_2(f_i)}_{n_i - terms}] \in CT_{(\tau)},$$

for all $\sigma_1, \sigma_2 \in Cohyp_G(\tau)$. So, $(Cohyp_G(\tau), \oplus_{CG})$ forms a semigroup.

Example 2.4. Let $\tau = (2)$ and $t = f[f[e_0^2, e_2^2], e_1^2],\ s = f[e_3^2, f[e_0^2, e_1^2]] \in CT_{(2)}$. Then

$$\begin{aligned}(\sigma_{f[f[e_0^2,e_2^2],e_1^2]} +_{CG} \sigma_{f[e_3^2,f[e_0^2,e_1^2]]})(f) &= \sigma_{f[e_3^2,f[e_0^2,e_1^2]]}(f)[\sigma_{f[f[e_0^2,e_2^2],e_1^2]}(f), \sigma_{f[f[e_0^2,e_2^2],e_1^2]}(f)]\\ &= f[e_3^2, f[e_0^2, e_1^2]][f[f[e_0^2, e_2^2], e_1^2], f[f[e_0^2, e_2^2], e_1^2]]\\ &= f[e_3^2, f[f[f[e_0^2, e_2^2], e_1^2], f[f[e_0^2, e_2^2], e_1^2]]], \text{ and}\end{aligned}$$

$$\begin{aligned}(\sigma_{f[f[e_0^2,e_2^2],e_1^2]} \oplus_{CG} \sigma_{f[e_3^2,f[e_0^2,e_1^2]]})(f) &= \sigma_{f[f[e_0^2,e_2^2],e_1^2]}(f)[\sigma_{f[e_3^2,f[e_0^2,e_1^2]]}(f), \sigma_{f[e_3^2,f[e_0^2,e_1^2]]}(f)]\\ &= f[f[e_0^2, e_2^2], e_1^2][f[e_3^2, f[e_0^2, e_1^2]], f[e_3^2, f[e_0^2, e_1^2]]]\\ &= f[f[f[e_3^2, f[e_0^2, e_1^2]], e_2^2], f[e_3^2, f[e_0^2, e_1^2]]].\end{aligned}$$

Throughout this paper, we denote:
σ_t := the generalized cohypersubstitution σ of type τ which maps f to the coterm t,
e_j^n := the injection symbol for all $0 \le j \le n-1, n \in \mathbb{N}$,
E := the set of all injection symbols, i.e., $E := \{e_j^n \mid n, j \in \mathbb{N}^+ := \mathbb{N} \cup \{0\}\}$,
$E(t)$:= the set of all injection symbols occuring in the coterm t.

3. Main Results

In this section, we focus on the set $Cohyp_G(2)$ of all generalized cohypersubstitutions of type $\tau = (2)$ with a binary operation "$+_{CG}$" on the set $Cohyp_G(2)$ defined by $(\sigma_1 +_{CG} \sigma_2)(f) := (\sigma_2(f))[\sigma_1(f), \sigma_1(f)]$ for all $\sigma_1, \sigma_2 \in Cohyp_G(2)$. Then we have $(Cohyp_G(2), +_{CG})$ is a semigroup. We describe idempotent and regular elements in $Cohyp_G(2)$. Firstly, we recall the definition of an idempotent element in the semigroup $(Cohyp_G(2), +_{CG})$.

Definition 3.1. Let $(Cohyp_G(2), +_{CG})$ be a semigroup. An element $\sigma_t \in Cohyp_G(2)$ is called idempotent if $\sigma_t +_{CG} \sigma_t = \sigma_t$. Denoted by $\mathcal{E}^{+_{CG}}(Cohyp_G(2))$ the set of all idempotent elements of $Cohyp_G(2)$.

Theorem 3.2. *Let $t, s \in CT_{(2)}$. Then the following statements hold.*

(i) If $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}$, then $t[s, s] = t$ if and only if $s = e_0^2$.

(ii) If $E(t) \cap \{e_0^2, e_1^2\} = \{e_1^2\}$, then $t[s, s] = t$ if and only if $s = e_1^2$.

(iii) If $E(t) \cap \{e_0^2, e_1^2\} = \emptyset$, then $t[s, s] = t$.

Proof. (i) Let $t, s \in CT_{(2)}$ where $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}$.

Let $t = f[t_1, t_2]$ where $t_1, t_2 \in CT_{(2)}$ and assume that $t[s, s] = t$. Suppose that $s \neq e_0^2$. Then

$$t[s, s] = (f[t_1, t_2])[s, s] = f[t_1[s, s], t_2[s, s]].$$

Since $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}$ and $s \neq e_0^2$, this force that $t_1[s, s] \neq t_1$ and $t_2[s, s] \neq t_2$. Thus $t[s, s] = (f[t_1, t_2])[s, s] = f[t_1[s, s], t_2[s, s]] \neq f[t_1, t_2] = t$, which is a contradiction. Hence, $s = e_0^2$.

Conversely, assume that $s = e_0^2$. We give a proof by indection on the complexity of the coterm t. If $t = e_0^2$, then $e_0^2[s, s] = e_0^2$. If $t = e_j^2$ for $j \geq 2$, then $e_j^2[s, s] = e_j^2$. If $t = f[t_1, t_2]$ and suppose that $t_1[s, s] = t_1$ and $t_2[s, s] = t_2$, then $t[s, s] = (f[t_1, t_2])[s, s] = f[t_1[s, s], t_2[s, s]] = f[t_1, t_2] = t$.

Similarly, we can proof (ii) and (iii). ∎

Theorem 3.3. *The generalized cohypersustitution σ_t of type $\tau = (2)$ is idempotent if and only if $t[\sigma_t(f), \sigma_t(f)] = t$.*

Proof. Let $t \in CT_{(2)}$. Assume that σ_t is an idempotent. Then
$t[\sigma_t(f), \sigma_t(f)] = (\sigma_t(f) +_{CG} \sigma_t)(f) = \sigma_t(f) = t$.

Conversely, assume that $t[\sigma_t(f), \sigma_t(f)] = t$. Then $(\sigma_t +_{CG} \sigma_t)(f) = (\sigma_t(f)[\sigma_t(f), \sigma_t(f)] = t[\sigma_t(f), \sigma_t(f)] = t = \sigma_t(f)$. Thus σ_t is an idempotent. ∎

Next, we study on the set of all projection generalized cohypersubstitutions of type $\tau = (2)$ which define as following.

Definition 3.4. Let $\tau = (2)$. A generalized cohypersubstitution σ of type $\tau = (2)$ is called a projection generalized cohypersubstitution if the coterm $\sigma(f_i)$ is the injection symbol for each $i \in I$. Let $\sigma_t \in P_{CG}^{inj}(2)$ be the set of all projection generalized cohypersubstitutions of type $\tau = (2)$, i.e., $\sigma_t \in P_{CG}^{inj}(2) := \{\sigma_{e^2,j} \mid e_j^2 \in E\}$.

By applying the Theorem 3.2 and Theorem 3.3, we have the following corollary.

Corollary 3.5. *Every $\sigma_t \in P_{CG}^{inj}(2)$ is idempotent.*

Corollary 3.6. *If $\sigma_t \in Cohyp_G(2)$ and $E(t) \cap \{e_0^2, e_1^2\} = \emptyset$, then σ_t is idempotent.*

Lemma 3.7. *Let $t_1, t_2 \in CT_{(2)}$ and $\sigma_t \in Cohyp_G(2)$. Then the following statements hold.*

(i) If $t = f[e_0^2, t_2]$ where $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}$, then σ_t is not idempotent.

(ii) If $t = f[t_1, e_1^2]$ where $E(t) \cap \{e_0^2, e_1^2\} = \{e_1^2\}$, then σ_t is not idempotent.

(iii) If $t = f[t_1, t_2]$ where $\{e_0^2, e_1^2\} \subseteq E(t)$, then σ_t is not idempotent.

Proof. (i) Let $\sigma_t \in Cohyp_G(2)$ where $t = f[e_0^2, t_2]$, $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}$ and $t_2 \in CT_{(2)}$.

Consider

$$\begin{aligned}
(\sigma_{f[e_0^2,t_2]} +_{CG} \sigma_{f[e_0^2,t_2]})(f) &= \sigma_{f[e_0^2,t_2]}(f)[\sigma_{f[e_0^2,t_2]}(f), \sigma_{f[e_0^2,t_2]}(f)] \\
&= f[e_0^2, t_2][f[e_0^2, t_2], f[e_0^2, t_2]] \\
&= f[f[e_0^2, t_2], t_2[f[e_0^2, t_2], f[e_0^2, t_2]]] \\
&\neq f[e_0^2, t_2].
\end{aligned}$$

Hence, σ_t is not idempotent.

Similarly, we can proof (ii).

(iii) Let $\sigma_t \in Cohyp_G(2)$ where $t = f[t_1, t_2]$, $t_1, t_2 \in CT_{(2)}$ and $\{e_0^2, e_1^2\} \subseteq E(t)$.

Consider

$$\begin{aligned}
(\sigma_{f[t_1,t_2]} +_{CG} \sigma_{f[t_1,t_2]})(f) &= \sigma_{f[t_1,t_2]}(f)[\sigma_{f[t_1,t_2]}(f), \sigma_{f[t_1,t_2]}(f)] \\
&= f[t_1, t_2][f[t_1, t_2], f[t_1, t_2]] \\
&= f[t_1[f[t_1, t_2], f[t_1, t_2]], t_2[f[t_1, t_2], f[t_1, t_2]]].
\end{aligned}$$

Since $\{e_0^2, e_1^2\} \subseteq E(t)$, then we have $t_1[f[t_1, t_2], f[t_1, t_2]] \neq t_1$ and $t_2[f[t_1, t_2], f[t_1, t_2]] \neq t_2$. So, $(\sigma_{f[t_1,t_2]} +_{CG} \sigma_{f[t_1,t_2]})(f) \neq \sigma_{f[t_1,t_2]}$. Hence, σ_t is not idempotent. ∎

Lemma 3.8. *Let $t_1, t_2 \in CT_{(2)}$ and $\sigma_t \in Cohyp_G(2)$. Then the following statements hold.*

(i) If $t = f[t_1, e_0^2]$ where $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}$, then σ_t is not idempotent.

(ii) If $t = f[e_1^2, t_2]$ where $E(t) \cap \{e_0^2, e_1^2\} = \{e_1^2\}$, then σ_t is not idempotent.

Proof. The proof of this lemma is similarl to Lemma 3.7. ∎

We now set $E^* := \{\sigma_t \mid E(t) \cap \{e_0^2, e_1^2\} = \emptyset\}$. So, we have the following theorem.

Theorem 3.9. $\mathcal{E}^{+_{CG}}(Cohyp_G(2)) := P_{CG}^{inj}(2) \cup E^*$ *is the set of all idempotent elements of* $(Cohyp_G(2), +_{CG})$.

Proof. The proof is directly optained from Corollary 3.5, Corollary 3.6, Lemma 3.7 and Lemma 3.8. ∎

By applying the method in [1], we have the following lemma.

Lemma 3.10. $\mathcal{E}^{+_{CG}}(Cohyp_G(2))$ *is a maximal idempotent subsemigroup of* $(Cohyp_G(2), +_{CG})$.

Proof. It is easy to see that $\mathcal{E}^{+_{CG}}(Cohyp_G(2)) \subset Cohyp_G(2)$ and it is closed under the operation $+_{CG}$. So, $\mathcal{E}^{+_{CG}}(Cohyp_G(2))$ is an idempotent subsemigroup of $(Cohyp_G(2), +_{CG})$. We next to show that it is a maximal idempotent subsemigroup.

Let $\mathcal{M}$ be a proper idempotent subsemigroup of $(Cohyp_G(2), +_{CG})$ such that $\mathcal{E}^{+_{CG}}(Cohyp_G(2)) \subseteq \mathcal{M} \subset Cohyp_G(2)$. Let $\sigma_t \in \mathcal{M}$. Then σ_t is an idempotent element. Suppose that $\sigma_t \neq \mathcal{E}^{+_{CG}}(Cohyp_G(2))$. Then, by Lemma 3.7 and Lemma 3.8, σ_t is not idempotent, which is a contradiction. So, $\sigma_t \in \mathcal{E}^{+_{CG}}(Cohyp_G(2))$. Hence, $\mathcal{M} = \mathcal{E}^{+_{CG}}(Cohyp_G(2))$.

Therefore, $\mathcal{E}^{+_{CG}}(Cohyp_G(2))$ is a maximal idempotent subsemigroup of $(Cohyp_G(2), +_{CG})$. ∎

Now, we will describe regular elements in the semigroup $(Cohyp_G(2), +_{CG})$.

Definition 3.11. Let $(Cohyp_G(2), +_{CG})$ be a semigroup. An element $\sigma_t \in Cohyp_G(2)$ is call regular if there exists $\sigma_s \in Cohyp_G(2)$ such that $\sigma_t +_{CG} \sigma_s +_{CG} \sigma_t = \sigma_t$. Denoted by $\mathcal{R}^{+_{CG}}(Cohyp_G(2))$ the set of all regular elements of $Cohyp_G(2)$.

Theorem 3.12. *For any type* $\tau = (2)$, $\mathcal{E}^{+_{CG}}(Cohyp_G(2)) = \mathcal{R}^{+_{CG}}(Cohyp_G(2))$.

Proof. Since every idempotent elements is regular element, so we have $\mathcal{E}^{+_{CG}}(Cohyp_G(2)) \subseteq \mathcal{R}^{+_{CG}}(Cohyp_G(2))$. We will show that $\mathcal{R}^{+_{CG}}(Cohyp_G(2)) = \mathcal{E}^{+_{CG}}(Cohyp_G(2))$. Let $\sigma_t \in \mathcal{R}^{+_{CG}}(Cohyp_G(2))$. Then there exists $\sigma_s \in Cohyp_G(2)$ such that $\sigma_t +_{CG} \sigma_s +_{CG} = \sigma_t$.

So,

$$
\begin{aligned}
(\sigma_t(f))[(\sigma_t +_{CG} \sigma_s(f))(f), (\sigma_t +_{CG} \sigma_s(f))(f)] &= \sigma_t(f) \\
t[s[t,t], s[t,t]] &= t.
\end{aligned}
$$

This force that $e_0^2, e_1^2 \notin E(t)$. Thus $\sigma_t \in E^*$.

Assume that $t \neq e_i^2; i \in \mathbb{N}^*$, then $s[t,t] \neq e_i^2; i \in \mathbb{N}^*$. So, by Theorem 3.2 (i),(ii), we obtain that $t[s[t,t], s[t,t]] \neq t$. We get a cotradiction. Thus $t = e_i^2; i \in \mathbb{N}^*$ which implies that $\sigma_t \in P_{CG}^{inj}(2)$. Hence $\sigma_t \in E^* \cup P_{CG}^{inj}(2) := \mathcal{E}^{+_{CG}}(Cohyp_G(2))$.

Therefore, $\mathcal{E}^{+_{CG}}(Cohyp_G(2)) = \mathcal{R}^{+_{CG}}(Cohyp_G(2))$. ∎

In the last of this section, we study on the set $Cohyp_G(2)$ of all generalized cohypersubstitutions of type $\tau = (2)$ together with a binary operation "$\oplus_{CG}$" on the set $Cohyp_G(2)$ defined by $(\sigma_1 \oplus_{CG} \sigma_2)(f) := (\sigma_1(f))[\sigma_2(f), \sigma_2(f)]$ for all $\sigma_1, \sigma_2 \in Cohyp_G(2)$. Then we have that $(Cohyp_G(2), \oplus_{CG})$ is a semigroup. We describe idempotent and regular elements in $Cohyp_G(2)$ by using the following definitions.

Definition 3.13. Let $(Cohyp_G(2), \oplus_{CG})$ be a semigroup. An element $\sigma_t \in Cohyp_G(2)$ is call idempotent if $\sigma_t \oplus_{CG} \sigma_t = \sigma_t$. Denoted by $\mathcal{E}^{\oplus_{CG}}(Cohyp_G(2))$ the set of all idempotent elements of $Cohyp_G(2)$.

Definition 3.14. Let $(Cohyp_G(2), \oplus_{CG})$ be a semigroup. An element $\sigma_t \in Cohyp_G(2)$ is call regular if there exists $\sigma_s \in Cohyp_G(2)$ such that $\sigma_t \oplus_{CG} \sigma_s \oplus_{CG} \sigma_t = \sigma_t$. Denoted by $\mathcal{R}^{\oplus_{CG}}(Cohyp_G(2))$ the set of all regular elements of $Cohyp_G(2)$.

We can see that every idempotent element in $(Cohyp_G(2), +_{CG})$ is idempotent element in $(Cohyp_G(2), \oplus_{CG})$ and also regular element. So, we have the following results.

Proposition 3.15. *For any type $\tau = (2)$, $\mathcal{E}^{\oplus_{CG}}(Cohyp_G(2)) = \mathcal{R}^{\oplus_{CG}}(Cohyp_G(2))$.*

Proof. The proof is similar to Theorem 3.12. ∎

So, we have the following corollary.

Corollary 3.16. *For any type $\tau = (2)$, $\mathcal{R}^{+_{CG}}(Cohyp_G(2)) = \mathcal{E}^{+_{CG}}(Cohyp_G(2)) = \mathcal{E}^{\oplus_{CG}}(Cohyp_G(2)) = \mathcal{R}^{\oplus_{CG}}(Cohyp_G(2))$.*

4. Conclusion

This study focuses on the semigroups $(Cohyp_G(2), +_{CG})$ and $(Cohyp_G(2), \oplus_{CG})$ of generalized cohypersubstitutions of type $\tau = (2)$. We characterize the idempotent and regular elements on these semigroups. The main results of the study shown that any regular elements are idempotent elements. Moreover, we can see that the set of all idempotent and regular elements of the semigroup $(Cohyp_G(2), +_{CG})$ equal to the set of all idempotent and regular elements of the semigroup $(Cohyp_G(2), \oplus_{CG})$.

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Shrinking Accelerated Forward-Backward-Forward Method for Split Equilibrium Problem and Monotone Inclusion Problem in Hilbert Spaces

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ABSTRACT

We propose and analyze a hybrid splitting method, comprises of forward-backward-forward iterates, shrinking projection iterates and Nesterov's acceleration method, to solve the monotone inclusion problem associated with maximal monotone operators and split equilibrium problem in Hilbert spaces. The proposed iterative method exhibits accelerated strong convergence characteristics under suitable set of control conditions in such framework. Finally, we explore some useful applications of the proposed iterative method via Numerical experiment.

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1. Introduction

The theory of convex optimization, as a subject, is developed to the discovery of problem arising in applied mathematics for which optimization algorithms or iterative methods are the effective tools. As a result, powerful optimization tools found valuable applications in core areas of applied mathematics as well as in automatic control systems, medicine, economics, signal processing, management, communications and networks, industry, combinatorial optimization, global optimization and other branches of sciences. Since convex optimization has a long historical roots, however, several recent developments in the subject not only stimulated the interest of researchers but also serve as an interdisciplinary bridge between various branches of sciences as mentioned above. It is therefore natural to recognize and formulate

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various real world and theoretical problems in the general framework of convex optimization to be solved numerically. Iterative methods are ubiquitous in the theory of convex optimization and still new iterative and theoretical techniques have been proposed and analyzed for convex optimization problems. Such an algorithm or iterative method is designed for the selection of the best out of many possible decisions in a real-life environment, constructing computational methods to find optimal solutions, exploring the theoretical properties and studying the computational performance of numerical algorithms implemented based on computational methods. Monotone operator theory is a fascinating field of research in nonlinear functional analysis. This class of operators attracts the research community primarily due to the importance of these operators in modelling problems in the field of convex optimization, subgradients, partial differential equations, variational inequalities and image processing, evolution equations and inclusions, see for instance, [12, 26, 30] and the references cited therein. Many problems in the theory of convex optimization concern with the approximation of zeroes of a maximal monotone operator defined on a Hilbert space. On the other hand, the problem of finding zeroes of the sum of two (maximal -) monotone operators is of fundamental importance in structured convex optimization, variational analysis, machine learning, signal processing and image analysis [22, 28]. Since the structured convex optimization problems are complex in nature and require sophisticated tools for the consequent analysis. Therefore, operator splitting technique is the most efficient tool to solve the structured convex optimization problem comprises of smooth and non-smooth functions. Moreover, operator splitting technique provides parallel computing architectures and thus reducing the complexity by splitting the original problem into simpler problems. The forward-backward (FB) algorithm is prominent among various splitting algorithms to find a zero of the sum of two maximal monotone operators [22]. Note that the FB algorithm efficiently tackle the situation for smooth and/or non-smooth functions. We remark that the several general splitting algorithms are available in the literature with specific limitations. However, new splitting algorithms are formulated in such a way to unify and/or combine the existing splitting algorithms with enhanced intrinsic properties. We, therefore, propose and analyze a splitting method comprises of forward-backward-forward (FBF) iterates in Hilbert spaces. In 1964, Polyak [29] employed the inertial extrapolation technique, based on the heavy ball methods of the two-order time dynamical system, to equip the iterative algorithm with fast convergence characteristic. It is remarked that the inertial term is computed by the difference of the two preceding iterations. The inertial extrapolation technique was originally proposed for minimizing differentiable convex functions, but it has been generalized in different ways. The heavy ball method has been incorporated in various iterative algorithms to obtain the fast convergence characteristic, see, for example [1] and the references cited therein. One of the main motivations for this paper is to equip the FBF algorithm with the inertial extrapolation technique for fast convergence results in Hilbert spaces. In order to ensure the strong convergence characteristics of the proposed algorithm, the shrinking effect of the half space is also employed in this framework. The theory of equilibrium problems is a systematic approach to study a diverse range of problems arising in the field of physics, optimization, variational inequalities, transportation, economics, network and noncooperative games, see, for example [13, 21] and the references cited therein. The existence result of an equilibrium problem can be found in the seminal work of Blum and Oettli [13]. Moreover, this theory has a computational flavor and flourishes significantly due to an excellent paper of Combettes and Hirstoaga [20]. The classical equilibrium problem theory has been generalized in several interesting ways to solve real world problems. In 2012, Censor et al. [18] proposed a theory regarding split variational inequality problem (SVIP) which aims to solve a pair of

variational inequality problem in such a way that the solution of a variational inequality problem, under a given bounded linear operator, solves another variational inequality.
Motivated by the work of Censor et al. [18], Moudafi [27] generalized the concept of SVIP to that of split monotone variational inclusions (SMVIP) which includes, as a special case, split variational inequality problem, split common fixed point problem, split zeroes problem, split equilibrium problem and split feasibility problem. These problems have already been studied and successfully employed as a model in intensity-modulated radiation therapy treatment planning, see [16,17]. This formalism is also at the core of modeling of many inverse problems arising for phase retrieval and other real-world problems; for instance, in sensor networks in computerized tomography and data compression; see, for example, [19]. Some methods have been proposed and analyzed to solve split equilibrium problem and mixed split equilibrium problem in Hilbert spaces, see, for example, [2–11] and the references cited therein. Inspired and motivated by the above mentioned results and the ongoing research in this direction, we aim to employ a hybrid splitting method, comprises of forward-backward-forward iterates, shrinking projection iterates and Nesterov's acceleration method, to solve the monotone inclusion problem associated with maximal monotone operators and split equilibrium problem in Hilbert spaces. The proposed iterative method exhibits accelerated strong convergence characteristics under suitable set of control conditions in such framework. Finally, we explore some useful applications of the proposed iterative method via numerical simulation. The rest of the paper is organized as follows: Section 2 contains preliminary concepts and results regarding monotone operator theory and equilibrium problem theory. Section 3 comprises of strong convergence results of the proposed algorithm. Section 4 deals with applications of (FBF) method in minimization problem, split feasibility problem, monotone variational inequality problem and Image processing. Section 5 deals with the efficiency of the proposed algorithm and its comparison with the existing algorithm by numerical experiments.

2. Preliminaries

Throughout this section, we first fix some necessary notions and concepts which will be required in the sequel(see [12] for a detailed account). We denote by $\mathbb{N}$ the set of all natural numbers and $\mathbb{R}$ the set of all real numbers, respectively. Let $C \subseteq \mathcal{H}_1$ and $Q \subseteq \mathcal{H}_2$ be two non-empty subsets of real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ with the inner product $\langle \cdot\, , \cdot \rangle$ and the associated norm $\|\cdot\|$. Let $x_n \to x$ (resp. $x_n \rightharpoonup x$) indicates strong convergence (resp. weak convergence) of a sequence $\{x_n\}_{n=1}^{\infty}$ in C. Let $A : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ be an operator. We denote $dom\,(A) = \{x \in \mathcal{H}_1 : Ax \neq \emptyset\}$ the domain of A, $Gr(A) = \{(x, u) \in \mathcal{H}_1 \times \mathcal{H}_1 : u \in Ax\}$ the graph of A and $zer\,(A) = \{x \in \mathcal{H}_1 : 0 \in Ax\}$ the set of zeros of A. The inverse of A, that is, A^{-1} is defined as $(u, x) \in Gr(A^{-1})$ if and only if $(x, u) \in Gr(A)$ and the resolvent of A is denoted as $J_A = (I + A)^{-1}$. It is remarked that $J_A : \mathcal{H}_1 \to \mathcal{H}_1$ is single valued and maximal monotone operator provided that A is maximal monotone. Recall that A is said to be: (i) monotone if $\langle x - y, u - v\rangle \geq 0$ for all $(x, u), (y, v) \in Gr(A)$; (ii) maximally monotone if A is monotone and there exists no monotone operatort $B : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ such that $Gr(B)$ properly contains the $Gr(A)$; (iii) strongly monotone with modulus $\alpha > 0$ such that $\langle x - y, u - v\rangle \geq \alpha \|x - y\|^2$ for all $(x, u), (y, v) \in Gr(A)$ and (iv) inverse strongly monotone (cocoercive) with parameter β such that $\langle x - y, Ax - Ay\rangle \geq \beta \|Ax - Ay\|^2$.

Let $f : \mathcal{H}_1 \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function and let $g : \mathcal{H}_1 \to \mathbb{R}$ be a convex, differentiable and Lipschitz continuous gradient function, then the convex minimization problem for f and g is defined as:

$$\min_{x\in\mathcal{H}_1}\{f(x)+g(x)\}. \tag{2.1}$$

The subdifferential of a function f is defined and denoted as:

$$\partial f(x)=\{x^*\in\mathcal{H}_1 : f(y)\geq f(x)+\langle x^*, y-x\rangle \text{ for all } x\in\mathcal{H}_1\}.$$

It is remarked that the subdifferential of a proper convex lower semicontinuous function is a maximally monotone operator. The proximity operator of a function f is defined as:

$$\operatorname{prox}_f:\mathcal{H}_1\to\mathcal{H}_1 : x\mapsto \operatorname*{argmin}_{y\in\mathcal{H}_1}\left(f(y)+\frac{1}{2}\|x-y\|^2\right).$$

Note that the proximity operator is linked with the subdifferential operator such that argmin $(f)=zer(\partial f)$. Moreover, $\operatorname{prox}_f=J_{\partial f}$. Utilizing the said connection, we state monotone inclusion problem with respect to a maximally monotone operator A and an arbitrary operator B is to find:

$$x^*\in C \quad \text{such that } 0\in Ax^*+Bx^*. \tag{2.2}$$

The solution set of the problem (2.1) is denoted as $zer(A+B)$.

We now define the concept of (mixed) split equilibrium problem.

Let $h:\mathcal{H}_1\to\mathcal{H}_2$ be a bounded linear operator. Let $F: C\times C\to\mathbb{R}$ and $G: Q\times Q\to\mathbb{R}$ be two bifunctions, $\phi_f: C\to\mathcal{H}_1$ and $\phi_g: Q\to\mathcal{H}_2$ be two nonlinear operators. Recall that a split equilibrium problem (SEP) is to find:

$$x^*\in C \quad \text{such that } F(x^*,x)\geq 0 \text{ for all } x\in C, \tag{2.3}$$

and

$$y^*=hx^*\in Q \quad \text{such that } G(y^*,y)\geq 0 \text{ for all } y\in Q. \tag{2.4}$$

The solution set of the split equilibrium problem (2.2) and (2.3) is denoted by

$$SEP(F):=\{x^*\in C : x^*\in EP(F) \quad \text{and} \quad hx^*\in EP(G)\}. \tag{2.5}$$

Let C be a nonempty closed convex subset of a Hilbert space $\mathcal{H}_1$. Let C be a nonempty closed convex subset of a Hilbert space $\mathcal{H}_1$. For each $x\in\mathcal{H}_1$, there exists a unique nearest point of C, denoted by P_Cx, such that

$$\|x-P_Cx\|\leq\|x-y\| \text{ for all } y\in C.$$

Such a mapping $P_C:\mathcal{H}_1\to C$ is known as a metric projection or a nearest point projection of $\mathcal{H}_1$ onto C. Moreover, P_C satisfies nonexpansiveness in a Hilbert space and $\langle x-P_Cx, P_Cx-y\rangle\geq 0$ for all $x,y\in C$. It is remarked that P_C is firmly nonexpansive mapping from $\mathcal{H}_1$ onto C, that is,

$$\|P_Cx-P_Cy\|^2\leq\langle x-y, P_Cx-P_Cy\rangle, \text{ for all } x,y\in C.$$

Moreover, for any $x\in\mathcal{H}_1$ and $z\in C$, we have $z=P_Cx$ if and only if $\langle x-z, z-y\rangle\geq 0$ for every $y\in C$. The following lemma collects some well-known results in the context of a real Hilbert space.

The following lemma collects some well-known results in the context of a real Hilbert space.

Lemma 2.1. [12] *The following properties hold in a real Hilbert space* $\mathcal{H}_1$*:*

1. $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$, *for all* $x, y \in \mathcal{H}_1$;
2. $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$, *for all* $x, y \in \mathcal{H}_1$;
3. $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$, *for every* $x, y \in \mathcal{H}_1$ *and* $\mu \in [0,1]$.

Lemma 2.2. [14] *Let* T *be an nonexpansive mapping with* $F(T) \neq \emptyset$ *defined it as for a closed nonempty convex subset* $\widehat{C}$ *of a real Hilbert space* $\mathcal{H}_1$*. we assume that* $\{u_n\}$ *is a sequence in* $\widehat{C}$ *such that* $u_n \rightharpoonup u$ *and* $(I-T)u_n \to v$*, then* $(I-T)u = v$*. In particular, if* $v = 0$*, then* $u \in F(T)$*.*

Asumption 2.3. *Let* C *be a nonempty closed and convex subset of a Hilbert space* $\mathcal{H}_1$*. Let* $F : C \times C \to \mathbb{R}$ *be a bifunction and lower semicontinuous satisfying the following conditions:*
(A1) $F(x,x) = 0$ *for all* $x \in C$;
(A2) F *is monotone, i.e.,* $F(x,y) + F(y,x) \leq 0$ *for all* $x, y \in C$;
(A3) *for each* $x, y, z \in C$, $\limsup_{t\to 0} F(tz + (1-t)x, y) \leq F(x,y)$;
(A4) *for each* $x \in C$, $y \mapsto F(x,y)$ *is convex and lower semi-continuous.*

Lemma 2.4. [24] *Let* C *be a closed convex subset of a real Hilbert space* $\mathcal{H}_1$ *and let* $F : C \times C \to \mathbb{R}$ *be a bifunction satisfying conditions (A1)-(A4) of Assumption 2.3. For* $r > 0$ *and* $x \in \mathcal{H}_1$*, there exists* $z \in C$ *such that*

$$F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \text{ for all } y \in C.$$

Moreover, define a mapping $T_r^F : \mathcal{H}_1 \to C$ *by*

$$T_r^F(x) = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad \text{for all } y \in C \right\},$$

for all $x \in \mathcal{H}_1$*. Then, the following results hold:*
(1) *for each* $x \in \mathcal{H}_1$, $T_r^F \neq \emptyset$;
(2) T_r^F *is single-valued;*
(3) T_r^F *is firmly nonexpansive, i.e.,*
for every $x, y \in \mathcal{H}_1, \left\|T_r^F x - T_r^F y\right\|^2 \leq \left\langle T_r^F x - T_r^F y, x-y\right\rangle$;
(4) $F(T_r^F) = SEP(F)$;
(5) $SEP(F)$ *is closed and convex.*

It is remarked that if $G : Q \times Q \to \mathbb{R}$ *is a bifunction satisfying conditions (A1)-(A4), wher* Q *is a nonempty closed and convex subset of a Hilbert space* $\mathcal{H}_2$*. Then for each* $s > 0$ *and* $w \in \mathcal{H}_2$ *we can define a mapping:*

$$T_s^G(w) = \left\{ d \in C : G(d,e) + \frac{1}{s}\langle e-d, d-w\rangle \geq 0, \quad \text{for all } e \in Q \right\},$$

which is, nonempty, single-valued and firmly nonexpansive. Then the following results hold:
(1) *for each* $w \in \mathcal{H}_2$, $T_s^G \neq \emptyset$;
(2) T_s^G *is single-valued;*
(3) T_s^G *is firmly nonexpansive;*
(4) $F(T_s^G) = SEP(G)$;
(5) $SEP(G)$ *is closed and convex.*

Lemma 2.5. [31] *Let E be a Banach space satisfying Opial's condition and let $\{x_n\}$ be a sequence in E. Let $l, m \in E$ be such that $\lim_{n\to\infty} \|x_n - l\|$ and $\lim_{n\to\infty} \|x_n - m\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to l and m, respectively, then $l = m$.*

Lemma 2.6. [23] *Let $A : E \to E$ be an γ-inverse strongly accretive of order r and $B : E \to 2^E$ an n-accretive operator,where E be a Banach space. Then the following inequalities holds:*

a) *For $c > 0$, $F(T_c^{A,B}) = (A+B)^{-1}(0)$.*

b) *For $0 < d \leq c$ and $u \in E, \|u - T_d^{A,B}u\| \leq 2\|u - T_c^{A,B}\|$.*

Lemma 2.7. [25] *Let $\widehat{C}$ be a closed nonempty and convex subset of a real Hilbert space $\mathcal{H}_1$. For every $r, s \in \mathcal{H}_1$ and $\gamma \in \mathbb{R}$ the set*
$D = \{u \in C : \|s - u\|^2 \leq \|r - u\|^2 + \langle z, u\rangle + \gamma\}$ is convex and closed.

Lemma 2.8. *Let $P_{\widehat{C}} : \mathcal{H} \to \widehat{C}$ be the metric projection from $\mathcal{H}$ onto $\widehat{C}$. Then the following inequality satisfied:*

$$\|v - P_{\widehat{C}}u\|^2 + \|u - P_{\widehat{C}}u\|^2 \leq \|u - v\|^2,$$

for all $u \in \mathcal{H}$ and for all $v \in \widehat{C}$ and $\widehat{C}$ be a closed nonempty and convex subset of a real Hilbert space $\mathcal{H}$.

3. Main results

In this section, we prove some strong convergence theorems of an inertial method with a forward-backward-forward splitting algorithm for solving the split equilibrium problem together with the monotone inclusion problem in the framework of Hilbert spaces. Now, We prove the following strong convergence Theorem.

Theorem 3.1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Let $F : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be two bifunctions satisfying Assumption 2.3 such that G is upper semicontinuous. Let $h : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator; let $A : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ be a maximally monotone operator and let $B : \mathcal{H}_1 \to \mathcal{H}_1$ be a monotone and ρ-Lipschitz operator for some $\rho > 0$. Assume that $\Gamma = (A+B)^{-1}(0) \cap \Omega \neq \emptyset$, where $\Omega := \{x^* \in C : x^* \in EP(F)$ and $hx^* \in EP(G)\}$, $\{\mu_n\} \subset (0, \frac{2}{\rho})$, $\{\gamma_n\} \subset [0, \gamma]$, $\gamma \in [0, \frac{1}{2})$, $\{r_n\} \subset (0, \infty)$, with $\alpha \in (0, \frac{1}{L})$ such that L is the spectral radius of h^*h and $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0, 1]$. For given $x_0, x_1 \in \mathcal{H}_1$ let the iterative sequence $\{x_n\}$,$\{m_n\}$,$\{u_n\}$,$\{v_n\}$ and $\{w_n\}$ be generated by*

$$\begin{cases} m_n = x_n + \gamma_n(x_n - x_{n-1}); \\ u_n = \beta_n m_n + (1-\beta_n)T_{r_n}^F(I - \alpha h^*(I - T_{r_n}^G)h)m_n; \\ v_n = J_{\mu_n A}(I - \mu_n B)u_n; \\ w_n = v_n - \mu_n(Bv_n - Bu_n); \\ C_{n+1} = \{\widehat{u} \in C_n : \|w_n - \widehat{u}\|^2 \leq \|x_n - \widehat{u}\|^2 - 2\gamma_n\langle x_n - \widehat{u}, x_{n-1} - x_n\rangle + \gamma_n^2\|x_{n-1} - x_n\|^2\}; \\ x_{n+1} = P_{C_n}(x_0). \end{cases} \tag{3.1}$$

Assume that the following conditions hold:

C1 $\sum_{n=1}^{\infty} \gamma_n\|x_n - x_{n-1}\| < \infty$;

C2 $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

C3 $\liminf_{n\to\infty} r_n > 0$;

C4 $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < \frac{2}{\rho}$.

Then the sequence $\{x_n\}$ generated by (3.1) strongly convergence to a point $\widehat{u} = P_\Gamma x_1$.

Proof. We solve this theorem by dividing it into six steps.
Step 1. Show that $P_{C_{n+1}}x_1$ is well-defined for every $x \in \mathcal{H}_1$. We know that $(A+B)^{-1}(0)$ and Ω are closed and convex by Lemma 2.4 and 2.6, respectively. From the definition of C_{n+1} and from Lemma 2.9 C_{n+1} is closed and convex for each $n \geq 1$. For each $n \in \mathbb{N}$ and let $\widehat{u} \in \Gamma$. Since we write $T_n = T^F_{r_n}(I - \alpha h^*(I - T^G_{r_n})h)$. Note that T_n is a quasi-nonexpansive mapping and $J_{\mu_n A}$ is nonexpansive. For simplicity of the algorithm, we write $w_n = \beta_n v_n + (1-\beta_n)T_n v_n$ and for every $n \in \mathbb{N}$. We have

$$\begin{aligned}
\|m_n - \widehat{u}\|^2 &= \|x_n - \widehat{u} - \gamma_n(x_{n-1} - x_n)\|^2 \\
&= \|x_n - \widehat{u}\|^2 + \gamma_n^2\|x_{n-1} - x_n\|^2 - 2\gamma_n\langle x_n - \widehat{u}, x_{n-1} - x_n\rangle. \qquad (3.2)
\end{aligned}$$

Further,

$$\begin{aligned}
\|u_n - \widehat{u}\|^2 &= \|\alpha_n m_n + (1-\alpha_n)T_n m_n - \widehat{u}\|^2 \\
&= \|\alpha_n(m_n - \widehat{u}) + (1-\alpha_n)(m_n - \widehat{u})\|^2 \\
&= \|m_n - \widehat{u}\|^2 \\
&= \|x_n - \widehat{u}\|^2 + \gamma_n^2\|x_{n-1} - x_n\|^2 - 2\gamma_n\langle x_n - \widehat{u}, x_{n-1} - x_n\rangle. \qquad (3.3)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\|v_n - \widehat{u}\|^2 &= \|J_{\mu_n A}(I - \mu_n B)u_n - J_{\mu_n A}(I - \mu_n B)\widehat{u}\|^2 \\
&= \|u_n - \mu_n Bu_n - (\widehat{u} - \mu_n B\widehat{u})\|^2 \\
&= \|u_n - \widehat{u} - (\mu_n Bu_n - \mu_n B\widehat{u})\|^2 \\
&\leq \|u_n - \widehat{u}\|^2 + \mu_n^2\|Bu_n - B\widehat{u})\|^2 - 2\mu_n\langle u_n - \widehat{u}, Bu_n - B\widehat{u}\rangle \\
&\leq \|u_n - \widehat{u}\|^2 + \mu_n^2\|Bu_n - B\widehat{u})\|^2 - \frac{2\mu_n}{\rho}\|Bu_n - B\widehat{u}\|^2 \\
&\leq \|u_n - \widehat{u}\|^2 + \mu_n(\mu_n - \frac{2}{\rho})\|Bu_n - B\widehat{u})\|^2 \\
&= \|u_n - \widehat{u}\|^2 \\
&\leq \|x_n - \widehat{u}\|^2 + \gamma_n^2\|x_{n-1} - x_n\|^2 - 2\gamma_n\langle x_n - \widehat{u}, x_{n-1} - x_n\rangle. \qquad (3.4)
\end{aligned}$$

Note that, if $\widehat{u} \in zer(A+B)$, then $(Bx_n)_{n\in\mathbb{N}}$ converges strongly to the unique dual solution

Bx [see proof: Theorem 26.14(ii) of [12]], so therefore, $Bu_n - B\widehat{u} \to 0$. So we observe that,

$$\begin{aligned}
\|w_n - \widehat{u}\|^2 &= \|v_n - \mu_n(Bv_n - Bu_n) - (\widehat{u} + B\widehat{u} - B\widehat{u})\|^2 \\
&= \|(v_n - \widehat{u}) - \mu_n(Bv_n - Bu_n) - (B\widehat{u} - B\widehat{u})\|^2 \\
&= \|(v_n - \widehat{u}) - \mu_n(Bv_n - B\widehat{u}) + \mu_n(Bu_n - B\widehat{u})\|^2 \\
&= \|(v_n - \widehat{u}) - \mu_n(Bv_n - B\widehat{u})\|^2 + \mu_n^2\|(Bu_n - B\widehat{u})\|^2 \\
&\quad +2\mu_n\langle(v_n - \widehat{u}) - \mu_n(Bv_n - B\widehat{u}), Bu_n - B\widehat{u}\rangle \\
&= \|(v_n - \widehat{u}) - \mu_n^2(Bv_n - B\widehat{u})\|^2 + \mu_n^2\|Bu_n - B\widehat{u}\|^2 \\
&\quad +2\mu_n\langle v_n - \widehat{u}, Bu_n - B\widehat{u}\rangle - 2\mu_n^2\langle Bv_n - B\widehat{u}, Bu_n - B\widehat{u}\rangle \\
&= \|(v_n - \widehat{u}) - \mu_n^2(Bv_n - B\widehat{u})\|^2 \\
&= \|v_n - \widehat{u}\|^2 + \mu_n^2\|Bv_n - B\widehat{u}\|^2 - 2\mu_n\langle v_n - \widehat{u}, Bv_n - B\widehat{u}\rangle \\
&= \|v_n - \widehat{u}\|^2 + \mu_n(\mu_n - \frac{2}{\rho})\|Bv_n - Bx\|^2 \\
&= \|v_n - \widehat{u}\|^2 \\
&= \|x_n - \widehat{u}\|^2 + \gamma_n^2\|x_{n-1} - x_n\|^2 - 2\gamma_n\langle x_n - \widehat{u}, x_{n-1} - x_n\rangle. \qquad (3.5)
\end{aligned}$$

$\widehat{u} \in C_n$, for all $n \geq 1$. Hence $\widehat{u} \in C_{n+1}$ implies $\Gamma \subset C_{n+1}$. Therefore $P_{C_{n+1}}x_1$ is well defined.
Step 2. Next we show that $\lim_{n\to\infty}\|x_n - \widehat{u}\|$ exists. Since Γ is nonempty, closed and convex subset of $\mathcal{H}_1$, there exist a unique $x^* \in \Gamma$ such that

$$x^* = P_\Gamma x_1.$$

From $P_{C_{n+1}}x_1$, $C_{n+1} \subset C_n$ and $x_{n+1} \in C_{n+1}$, for all $n \geq 1$, we get

$$\|x_n - \widehat{u}\| \leq \|x_{n+1} - \widehat{u}\|, \text{for all } n \geq 1.$$

In either case, as $\Gamma \subset C_n$, we have

$$\|x_n - \widehat{u}\| \leq \|x^* - \widehat{u}\|, \text{for all } n \geq 1. \qquad (3.6)$$

This implies that $\{x_n\}$ is bounded, nondecreasing and well defined, hence

$$\lim_{n\to\infty}\|x_n - \widehat{u}\| \qquad \text{exists.} \qquad (3.7)$$

Step 3. Next, we show that $x_n \to \widehat{u} \in C$ as $n \to \infty$. For $m > n$, by the definition of C_n, we have $x_m = P_{C_m}x_1 \in C_m \subseteq C_n$. By Lemma 2.8 we estimate that,

$$\|x_m - x_n\|^2 \leq \|x_m - \widehat{u}\|^2 - \|x_n - \widehat{u}\|^2. \qquad (3.8)$$

Since, $\lim_{n\to\infty}\|x_n - \widehat{u}\|$ exists, it follows from (3.8) that $\lim_{n\to\infty}\|x_m - x_n\| = 0$. Hence, $\{x_n\}$ is a Cauchy sequence in C and $x_n \to \widehat{u} \in C$ as $n \to \infty$.
Step 4. Show that $\lim_{n\to\infty} x_n = \widehat{u}$, where $\widehat{u} = P_\Gamma(x_0)$. we obtain from (3.1) and Step 3 that, let $\widehat{u} \in zer(A + B)$, that is $-B\widehat{u} \in A\widehat{u}$. According to the definition of the resolvent, we have

$$\|m_n - x_n\| = |\gamma_n|\|x_n - x_{n-1}\| = 0. \qquad (3.9)$$

as $n \to +\infty$, $\|x_n - x_{n-1}\| \to 0$.

From (3.1), we get,

$$\begin{aligned}\|u_n - x_n\|^2 &= \|\alpha_n m_n + (1-\alpha_n)T_n m_n - x_n\|^2 \\ &\leq \|\alpha_n(m_n - x_n) + (1-\alpha_n)(m_n - x_n)\|^2 \\ &\leq \|m_n - x_n\|^2. \end{aligned} \tag{3.10}$$

Hence,

$$\|x_{n+1} - m_n\| \leq \|x_{n+1} - x_n\| + \|m_n - x_n\| \to 0 \quad as \quad n \to \infty.$$

Since, $x_{n+1} \in C_n$, we obtain,

$$\begin{aligned}\|v_n - x_{n+1}\|^2 &\leq \|x_n - x_{n+1}\|^2 - 2\gamma_n\langle x_n - x_{n+1}, x_{n-1} - x_n\rangle + \gamma_n^2\|x_{n-1} - x_n\|^2 \\ &\leq \|x_n - x_{n+1}\|^2 + 2|\gamma_n|\|x_n - x_{n+1}\|\|x_{n-1} - x_n\| \\ &\quad + \gamma_n^2\|x_{n-1} - x_n\|^2 \to 0, \quad as \quad n \to \infty. \end{aligned} \tag{3.11}$$

for all $n \geq 1$. So,

$$\lim_{n\to\infty}\|v_n - x_n\| = 0. \tag{3.12}$$

and

$$\begin{aligned}\|w_n - x_n\|^2 &= \|v_n - x_n - \mu_n(Bv_n - Bu_n)\|^2 \\ &\leq \|v_n - x_n\|^2 + \mu_n^2\|Bv_n - Bu_n\|^2 - \frac{2\mu_n}{\rho}\|Bv_n - Bu_n\|^2 \\ &\leq \|v_n - x_n\|^2 + \mu_n(\mu_n - \frac{2}{\rho})\|Bv_n - Bu_n\|^2 \\ &\leq \|v_n - x_n\|^2 \to 0, asn \to \infty. \end{aligned} \tag{3.13}$$

Similarly,

$$\|w_n - v_n\| \leq \|w_n - x_n\| + \|v_n - x_n\| \to 0 \quad as \quad n \to \infty.$$

$$\|v_n - u_n\| \leq \|v_n - x_n\| + \|u_n - x_n\| \to 0 \quad as \quad n \to \infty.$$

$$\|u_n - m_n\| \leq \|u_n - x_n\| + \|m_n - x_n\| \to 0 \quad as \quad n \to \infty.$$

and

$$\|w_n - m_n\| \leq \|w_n - m_n\| + \|m_n - x_n\| \to 0 \quad as \quad n \to \infty. \tag{3.14}$$

Take $w_n := S_n m_n$, where $S_n := (I + \mu_n A)^{-1}(I - \mu_n B)$. Therefore,

$$\|S_n m_n - m_n\| = \|w_n - m_n\| \to 0 \quad as \quad n \to \infty.$$

Since $\liminf_{n\to\infty}\mu_n > 0$, there exist $\sigma > 0$ such that $\mu_n \geq \sigma$, for all $n \geq 1$. Then, by Lemma 2.6, we have,

$$\lim_{n\to\infty}\|S_\sigma m_n - m_n\| \leq 2\lim_{n\to\infty}\|S_n m_n - m_n\| = 0.$$

By lemma 3.3 and 3.1 of [23], S_σ is nonexpansive and $F(S_\sigma) = (A+B)^{-1}(0)$. Since $\{x_n\}$ is bounded and $\mathcal{H}_1$ is reflexive, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \widehat{w} \in \mathcal{H}_1$. Using the fact that $\|u_n - x_n\| \to 0, \quad as \quad n \to \infty$ and $x_{n_i} \rightharpoonup \widehat{w} \in \mathcal{H}_1$, we have $u_{n_i} \rightharpoonup \widehat{w} \in \mathcal{H}_1$.

We can therefore make use of Lemma 2.2 to assure that $\widehat{w} \in \Gamma$. If $\widehat{u} = P_\Gamma(x_0)$, it follows from (3.7), the fact that $\widehat{w} \in \Gamma$ and the lower semicontinuity of the norm that,

$$\begin{aligned} \|x_0 - \widehat{u}\| &\leq \|x_0 - \widehat{w}\| \leq \liminf_{i\to\infty} \|x_0 - x_{n_i}\| \\ &\leq \limsup_{i\to\infty} \|x_0 - x_{n_i}\| \leq \|x_0 - \widehat{u}\|. \end{aligned}$$

Thus, we have that $\lim_{i\to\infty} \|x_{n_i} - x_0\| = \|x_0 - \widehat{w}\| = \|x_0 - \widehat{u}\|$.This implies that $x_{n_i} \to \widehat{w} = \widehat{u}, i \to \infty$. It follows that $\{x_n\}$ converges weakly to $\widehat{u}$. So we have,

$$\begin{aligned} \|x_0 - \widehat{u}\| &\leq \liminf_{n\to\infty} \|x_0 - x_n\| \\ &\leq \limsup_{n\to\infty} \|x_0 - x_n\| \leq \|x_0 - \widehat{u}\|. \end{aligned}$$

This shows that, $\lim_{n\to\infty} \|x_n - x_0\| = \|x_0 - \widehat{u}\|$. From $x_n \rightharpoonup \widehat{u}$, we also have $x_n - x_0 \rightharpoonup \widehat{u} - x_0$. Since $\mathcal{H}_1$ satisfies the Kadec-Klee property, it follows that $x_n - x_0 \to \widehat{u} - x_0$.
Therefore, $x_n \to \widehat{u}$ *as* $n \to \infty$.

Step 5. First we show that $h^*(I - T^G_{r_n})h$ is a $\frac{1}{L}$-inverse strongly monotone mapping. For this, we utilize the firmly nonexpansive of $T^G_{r_n}$ which implies that $(I - T^G_{r_n})$ is a 1-inverse strongly monotone mapping. Now, observe that

$$\begin{aligned} \|h^*(I - T^G_{r_n})hx - h^*(I - T^G_{r_n})hy\|^2 &= \langle h^*(I - T^G_{r_n})(hx - hy), h^*(I - T^G_{r_n})(hx - hy)\rangle \\ &= \langle (I - T^G_{r_n})(hx - hy), h^*h(I - T^G_{r_n})(hx - hy)\rangle \\ &\leq L\langle (I - T^G_{r_n})(hx - hy), (I - T^G_{r_n})(hx - hy)\rangle \\ &= L\|(I - T^G_{r_n})(hx - hy)\|^2 \\ &\leq L\langle x - y, h^*(I - T^G_{r_n})(hx - hy)\rangle, \end{aligned}$$

for all $x, y \in \mathcal{H}_1$. So, we observe that, $h^*(I - T^G_{r_n})h$ is a $\frac{1}{L}$-inverse strongly monotone. Moreover,$I - \alpha h^*(I - T^G_{r_n})h$ is nonexpansive provided $\alpha \in (0, \frac{1}{L})$.

Next, we show that $\widehat{v} \in \Omega$. Setting, $z_n = T^F_{r_n}(I - \alpha h^*(I - T^G_{r_n})h)m_n$. For any $\widehat{u} \in \Gamma$, we consider the following estimate:

$$\begin{aligned} \|z_n - \widehat{u}\|^2 &= \|T^F_{r_n}(I - \alpha h^*(I - T^G_{r_n})h)m_n - \widehat{u}\|^2 \\ &= \|T^F_{r_n}(I - \alpha h^*(I - T^G_{r_n})h)m_n - T^F_{r_n}\widehat{u}\|^2 \\ &\leq \|m_n - \alpha h^*(I - T^G_{r_n})hm_n - \widehat{u}\|^2 \\ &\leq \|m_n - \widehat{u}\|^2 + \alpha^2\|h^*(I - T^G_{r_n})hm_n\|^2 \\ &\quad +2\alpha\langle \widehat{u} - m_n, h^*(I - T^G_{r_n})hm_n\rangle. \end{aligned} \tag{3.15}$$

Thus, we have

$$\begin{aligned} \|z_n - \widehat{u}\|^2 \leq\ & \|x_n - \widehat{u}\|^2 + \gamma_n^2\|x_{n-1} - x_n\|^2 - 2\gamma_n\langle x_n - u, x_{n-1} - x_n\rangle \\ & +\alpha^2\langle hm_n - T^G_{r_n}hm_n, h^*h(I - T^G_{r_n})hm_n\rangle \\ & +2\alpha\langle \widehat{u} - m_n, h^*(I - T^G_{r_n})hm_n\rangle. \end{aligned} \tag{3.16}$$

Moreover, we have

$$\begin{aligned}\alpha^2\langle hm_n - T_{r_n}^G hm_n, h^*h(I - T_{r_n}^G)hm_n\rangle &\leq L\alpha^2\langle hm_n - T_{r_n}^G hm_n, hm_n - T_{r_n}^G hm_n\rangle \\ &= L\alpha^2\|hm_n - T_{r_n}^G hm_n\|^2. \qquad (3.17)\end{aligned}$$

Note that

$$\begin{aligned}2\alpha\langle \widehat{u} - m_n, h^*(I - T_{r_n}^G)hm_n\rangle &= 2\alpha\langle h(\widehat{u} - m_n), hm_n - T_{r_n}^G hm_n\rangle \\ &= 2\alpha\langle h(\widehat{u} - m_n), (hm_n - T_{r_n}^G hm_n) \\ &\quad -(hm_n - T_{r_n}^G hm_n), hm_n - T_{r_n}^G hm_n\rangle \\ &= 2\alpha[\langle Ap - T_{r_n}^G hm_n, hm_n - T_{r_n}^G hm_n\rangle - \|hm_n - T_{r_n}^G hm_n\|^2] \\ &\leq 2\alpha[\frac{1}{2}\|hm_n - T_{r_n}^G hm_n\|^2 - \|hm_n - T_{r_n}^G hm_n\|^2] \\ &= -\alpha\|hm_n - T_{r_n}^G hm_n\|^2. \qquad (3.18)\end{aligned}$$

Utilizing (3.16)-(3.18), we have

$$\begin{aligned}\|z_n - \widehat{u}\|^2 &\leq \|x_n - \widehat{u}\|^2 + \gamma_n^2\|x_{n-1} - x_n\|^2 - 2\gamma_n\langle x_n - u, x_{n-1} - x_n\rangle \\ &\quad +L\alpha^2\|hm_n - T_{r_n}^G hm_n\|^2 - \alpha\|hm_n - T_{r_n}^G hm_n\|^2 \\ &= \|x_n - \widehat{u}\|^2 + \gamma_n^2\|x_{n-1} - x_n\|^2 - 2\gamma_n\langle x_n - u, x_{n-1} - x_n\rangle \\ &\quad +\alpha(L\alpha - 1)\|hm_n - T_{r_n}^G hm_n\|^2. \qquad (3.19)\end{aligned}$$

Note that

$$\begin{aligned}\|u_n - \widehat{u}\|^2 &\leq \beta_n\|m_n - \widehat{u}\|^2 + (1 - \beta_n)\|z_n - \widehat{u}\|^2 \\ &\leq \|x_n - \widehat{u}\|^2 + \gamma_n^2\|x_{n-1} - x_n\|^2 - 2\gamma_n\langle x_n - u, x_{n-1} - x_n\rangle \\ &\quad +\alpha(L\alpha - 1)\|hm_n - T_{r_n}^G hm_n\|^2. \qquad (3.20)\end{aligned}$$

Moreover, we have

$$\begin{aligned}-\alpha(L\alpha - 1)\|hm_n - T_{r_n}^G hm_n\|^2 &\leq \|x_n - \widehat{u}\|^2 - \|u_n - \widehat{u}\|^2 + \gamma_n^2\|x_{n-1} - x_n\|^2 \\ &\quad -2\gamma_n\langle x_n - u, x_{n-1} - x_n\rangle. \qquad (3.21)\end{aligned}$$

Since $\alpha(L\alpha - 1) < 0$, it follows from (3.7), C1 and the above estimate that

$$\lim_{n\to\infty} \|hm_n - T_{r_n}^G hm_n\| = 0. \qquad (3.22)$$

Note that $T_{r_n}^F$ is firmly nonexpansive and $I-\alpha h^*(I-T_{r_n}^G)h$ is nonexpansive, it follows that

$$\begin{aligned}
\|z_n-\widehat{u}\|^2 &= \|T_{r_n}^F(m_n-\alpha h^*(I-T_{r_n}^G)hm_n)-T_{r_n}^F\widehat{u}\|^2\\
&\le \langle T_{r_n}^F(m_n-\alpha h^*(I-T_{r_n}^G)hm_n)-T_{r_n}^F\widehat{u}, m_n-\alpha h^*(I-T_{r_n}^G)hm_n-\widehat{u}\rangle\\
&= \langle z_n-\widehat{u}\ ,\ m_n-\alpha h^*(I-T_{r_n}^G)hm_n-\widehat{u}\rangle\\
&= \frac{1}{2}\{\|z_n-\widehat{u}\|^2+\|m_n-\alpha h^*(I-T_{r_n}^G)hm_n-\widehat{u}\|^2\\
&\quad -\|z_n-m_n+\alpha h^*(I-T_{r_n}^G)hm_n\|^2\}\\
&\le \frac{1}{2}\{\|z_n-\widehat{u}\|^2+\|m_n-\widehat{u}\|^2-\|z_n-m_n+\alpha h^*(I-T_{r_n}^G)hm_n\|^2\}\\
&= \frac{1}{2}\{\|z_n-\widehat{u}\|^2+\|m_n-\widehat{u}\|^2-(\|z_n-m_n\|^2+\alpha^2\|h^*(I-T_{r_n}^G)hm_n\|^2\\
&\quad +2\alpha\langle z_n-m_n\ ,\ h^*(I-T_{r_n}^G)hm_n\rangle)\}.
\end{aligned}$$

So, we have

$$\begin{aligned}
\|z_n-\widehat{u}\|^2 &\le \|m_n-\widehat{u}\|^2-\|z_n-m_n\|^2-2\alpha\langle z_n-m_n, h^*(I-T_{r_n}^G)hm_n\rangle\\
&\le \|m_n-\widehat{u}\|^2-\|z_n-m_n\|^2+2\alpha\|z_n-m_n\|\|h^*(I-T_{r_n}^G)hm_n\|. \qquad (3.23)
\end{aligned}$$

This implies that

$$\begin{aligned}
\|u_n-\widehat{u}\|^2 &\le \beta_n\|m_n-\widehat{u}\|^2+(1-\beta_n)\|z_n-\widehat{u}\|^2\\
&\le \beta_n\|m_n-\widehat{u}\|^2+(1-\beta_n)(\|m_n-\widehat{u}\|^2-\|z_n-m_n\|^2\\
&\quad +2\alpha\|z_n-m_n\|\|h^*(I-T_{r_n}^G)hm_n\|).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
(1-\beta_n)\|z_n-m_n\|^2 &\le \|m_n-\widehat{u}\|^2-\|u_n-\widehat{u}\|^2\\
&\quad +2\alpha(1-\beta_n)\|z_n-m_n\|\|h^*(I-T_{r_n}^G)hm_n\|. \qquad (3.24)
\end{aligned}$$

Utilizing (3.2-3.3), (3.22) and (C2), we have

$$\lim_{n\to\infty}\|z_n-m_n\|=0. \qquad (3.25)$$

From (3.9) and (3.25), we have

$$\begin{aligned}
\lim_{n\to\infty}\|z_n-m_n+x_n-x_n\| &= 0\\
\lim_{n\to\infty}\|z_n-x_n-(m_n-x_n)\| &= 0\\
\lim_{n\to\infty}\|z_n-x_n\| &= 0. \qquad (3.26)
\end{aligned}$$

Letting $n\to 0$ implies that $z_n \rightharpoonup \widehat{v}$. Next, we show that $\widehat{v}\in EP(F)$. Since, $z_n=T_{r_n}^F(I-\alpha h^*(I-T_{r_n}^G)h)m_n$, for any $y\in C$, we have

$$F(z_n,y)+\frac{1}{r_n}\langle y-z_n, z_n-x_n-\alpha h^*(I-T_{r_n}^G hm_n)\rangle\ge 0.$$

This implies that

$$F(z_n, y) + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle - \frac{1}{r_n}\langle y - z_n, \alpha h^*(I - T^G_{r_n} hm_n)\rangle \geq 0.$$

From the Assumption 2.3(A2), we have

$$\frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle - \frac{1}{r_n}\langle y - z_n, \alpha h^*(I - T^G_{r_n} hm_n)\rangle \geq -F(z_n, y) \geq F(\widehat{u}, z_n).$$

So, we have

$$\frac{1}{r_{n_i}}\langle y - z_{n_i}, z_{n_i} - x_{n_i}\rangle - \frac{1}{r_{n_i}}\langle y - z_{n_i}, \alpha h^*(I - T^G_{r_n} hm_{n_i})\rangle \geq F(y, z_{n_i}). \tag{3.27}$$

Utilizing (3.22) and (C3), we get that $z_{n_i} \rightharpoonup \widehat{v}$. Moreover, utilizing (3.34) and the Assumption 2.3(A2), we estimate

$$F(y, \widehat{v}) \leq 0, \quad \text{for all} \quad y \in C.$$

Let $y_t = ty + (1-t)\widehat{v}$ for some $1 \geq t > 0$ and $y \in C$. Since $\widehat{v} \in C$, consequently, $y_t \in C$ and hence $F(y_t, \widehat{v}) \leq 0$. Using Assumption 2.3((A1) and (A4)), it follows that

$$\begin{aligned} 0 &= F(y_t, y_t) \\ &\leq tF(y_t, y) + (1-t)F(y_t, \widehat{v}) \\ &\leq t(F(y_t, y)). \end{aligned}$$

This implies that

$$F(y_t, y) \geq 0, \quad \text{for all } y \in C.$$

Letting $t \to 0$ and by, Assumption 2.3 (A3), we get

$$F(\widehat{v}, y) \geq 0, \quad \text{for all } y \in C.$$

Thus, $\widehat{v} \in EP(F)$. Similarly, we can show that $\widehat{v} \in EP(G)$. Since h is a bounded linear operator, we have $hx_{n_i} \rightharpoonup h\widehat{v}$. It follow from (3.22) that

$$T^G_{r_{n_i}} hm_{n_i} \rightharpoonup h\widehat{v} \quad as \;\; i \to \infty. \tag{3.28}$$

Now, from Lemma 2.4 we have

$$G(T^G_{r_{n_i}} hm_{n_i}, y) + \frac{1}{r_{n_i}}\langle y - T^G_{r_{n_i}} hm_{n_i}, T^G_{r_{n_i}} hm_{n_i} - hm_{n_i}\rangle \geq 0,$$

for all $y \in C$. Since G is upper semicontinuous in the first argument and from (3.28), we have

$$G(h\widehat{v}, y) \geq 0,$$

for all $y \in C$. This implies that $h\widehat{v} \in EP(G)$. Therefore, $\widehat{v} \in SEP(F, G)$ and hence $\widehat{v} \in \Gamma$. This completes the proof.

∎

If we take $G = 0$ then split equilibrium problem goes to the classical equilibrium problem. So from (3.1), we get

Corollary 3.2. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a bifunctions satisfying (A1)-(A4) of Assumption 2.3. Let $A : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ be a maximally monotone operator and let $B : \mathcal{H}_1 \to \mathcal{H}_1$ be a monotone and ρ-Lipschitz operator for some $\rho > 0$. Assume that $\Gamma = (A+B)^{-1}(0) \cap \Omega \neq \emptyset$, where $\Omega := \{x^* \in C : x^* \in EP(F)\}$. For given $x_0, x_1 \in \mathcal{H}_1$ let the iterative sequence $\{x_n\}$,$\{m_n\}$,$\{u_n\}$,$\{v_n\}$ and $\{w_n\}$ are generated by*

$$\begin{cases} m_n = x_n + \gamma_n(x_n - x_{n-1}); \\ u_n = \beta_n m_n + (1-\beta_n) T^F_{r_n} m_n; \\ v_n = J_{\mu_n A}(I - \mu_n B)u_n; \\ w_n = v_n - \mu_n(Bv_n - Bu_n); \\ C_{n+1} = \{\widehat{u} \in C_n : \|w_n - \widehat{u}\|^2 \leq \|x_n - \widehat{u}\|^2 - 2\gamma_n\langle x_n - \widehat{u}, x_{n-1} - x_n\rangle + \gamma_n^2\|x_{n-1} - x_n\|^2\}; \\ x_{n+1} = P_{C_n}(x_0). \end{cases} \tag{3.29}$$

Let a sequence $\{x_n\}_{n=0}^{\infty}$ in $\mathcal{H}$ be generated by (3.29), for each $n \geq 1$, where $\{\mu_n\} \subset (0, 2\rho)$, $\{\gamma_n\} \subset [0, \gamma]$, $\gamma \in [0, \frac{1}{2})$, $\{r_n\} \subset (0, \infty)$ and $\{\beta_n\}$ are sequence in $[0,1]$. Assume that the following conditions hold:

C1 $\sum_{n=1}^{\infty} \gamma_n\|x_n - x_{n-1}\| < \infty$;

C2 $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

C3 $\liminf_{n\to\infty} r_n > 0$;

C4 $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < \frac{2}{\rho} < 2\rho$.

Then the sequence $\{x_n\}$ generated by (3.29) strongly convergence to a point $\widehat{u} = P_\Gamma x_1$.

If we take $B := 0$ in (3.1), then we obtain the following corollary,

Corollary 3.3. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Let $F : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be two bifunctions satisfying Assumption 2.3 such that G is upper semicontinuous. Let $h : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator; let $A : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ be a maximally monotone operator. Assume that $\Gamma = (A)^{-1}(0) \cap \Omega \neq \emptyset$, where $\Omega := \{x^* \in C : x^* \in EP(F) \text{ and } hx^* \in EP(G)\}$. For given $x_0, x_1 \in \mathcal{H}_1$ let the iterative sequence $\{x_n\}$,$\{m_n\}$,$\{u_n\}$,$\{v_n\}$ and $\{w_n\}$ are generated by*

$$\begin{cases} m_n = x_n + \gamma_n(x_n - x_{n-1}); \\ u_n = \beta_n m_n + (1-\beta_n) T^F_{r_n}(I - \alpha h^*(I - T^G_{r_n})h)m_n; \\ v_n = J_{\mu_n A} u_n; \\ w_n = v_n; \\ C_{n+1} = \{\widehat{u} \in C_n : \|w_n - \widehat{u}\|^2 \leq \|x_n - \widehat{u}\|^2 - 2\gamma_n\langle x_n - \widehat{u}, x_{n-1} - x_n\rangle + \gamma_n^2\|x_{n-1} - x_n\|^2\}; \\ x_{n+1} = P_{C_n}(x_0). \end{cases} \tag{3.30}$$

Let a sequence $\{x_n\}_{n=0}^{\infty}$ in $\mathcal{H}$ be generated by (3.30), for each $n \geq 1$, where $\{\mu_n\} \subset (0, 2\rho)$, $\{\gamma_n\} \subset [0, \gamma]$, $\gamma \in [0, \frac{1}{2})$, $\{r_n\} \subset (0, \infty)$, with $\alpha \in (0, \frac{1}{L})$ such that L is the spectral radius of h^*h and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$. Assume that the following conditions hold:

C1 $\sum_{n=1}^{\infty} \gamma_n\|x_n - x_{n-1}\| < \infty$;

C2 $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

C3 $\liminf_{n\to\infty} r_n > 0$;

C4 $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < 2\rho$.

Then the sequence $\{x_n\}$ generated by (3.30) strongly convergence to a point $\widehat{u} = P_\Gamma x_1$.

Remark 3.4. we remark here that the condition C1, it is easily applicable in numerical calculation since the valued of $\|x_n - x_{n-1}\|$ is known before choosing γ_n. At here, the parameter γ_n can be taken as $0 \leq \gamma_n \leq \widehat{\gamma_n}$,

$$\widehat{\gamma_n} = \begin{cases} \min\{\frac{w_n}{\|x_n - x_{n-1}\|}, \gamma\} & \text{if } x_n \neq x_{n-1}; \\ \gamma & \text{othrwise,} \end{cases}$$

where $\{w_n\}$ is a positive sequence such that $\sum_{n=1}^{\infty} w_n < \infty$ and $\gamma \in [0, 1)$.

4. Applications

In this section, we illustrate the theoretical results which we already obtained in previous section.

Convex Minimization Problem:

Let $\mathfrak{f} : \mathcal{H} \to \mathbb{R}$ and $\mathfrak{g} : \mathcal{H} \to \mathbb{R}$ be two convex, proper and lower semicontinuous functions such that a function $\mathfrak{f}$ and its differentiable with L-Lipschitz continuous gradient and another one function $\mathfrak{g}$ which is sub-differential and it is easily calculated. Assume that ω is the set of solutions of problem (2.1) and $\omega \neq 0$. In theorem 3.3, set that $B := \nabla\mathfrak{f}$ and $A := \partial\mathfrak{g}$. Then, we compute the following theorem for solving (2.1) in strong convergence with inertial and split equilibrium problem form.

Corollary 4.1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Let $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ be two bifunctions satisfying (A1)-(A4) of Assumption 2.3 such that G is upper semicontinuous. Let $h : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Let $A : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ be a maximally monotone operator and let $B : \mathcal{H}_1 \to \mathcal{H}_1$ be a monotone and ρ-Lipschitz operator for some $\rho > 0$. Let $\mathfrak{f}, \mathfrak{g} : \mathcal{H}_1 \to \mathbb{R}$ be two convex, proper and lower semicontinuous functions, such that a function $\mathfrak{f}$ which is differentiable with ρ-Lipschitz continuous gradient and another function $\mathfrak{g}$ which is sub-differential and it is easily calculated. Assume that ω is the set of solutions of problem (2.1) and $\omega \neq 0$. Let γ_n be a bounded real sequence and $\mu \in (0, \frac{2}{\rho})$. Assume that $\Gamma = \omega \cap \Omega \neq \emptyset$, where $\Omega := \{x^* \in C : x^* \in EP(F) \quad \text{and} \quad hx^* \in EP(G)\}$. For given $x_0, x_1 \in \mathcal{H}_1$ let the iterative sequence $\{x_n\}$,$\{m_n\}$,$\{u_n\}$,$\{v_n\}$ and $\{w_n\}$ are generated by*

$$\begin{cases} m_n = x_n + \gamma_n(x_n - x_{n-1}); \\ u_n = \beta_n m_n + (1 - \beta_n)T^F_{r_n}(I - \alpha h^*(I - T^G_{r_n})h)m_n; \\ v_n = prox_{\mu_n \mathfrak{g}}(I - \mu_n \nabla\mathfrak{f})u_n; \\ w_n = v_n - \mu_n(\nabla\mathfrak{f}v_n - \nabla\mathfrak{f}u_n); \\ C_{n+1} = \{\widehat{u} \in C_n : \|w_n - \widehat{u}\|^2 \leq \|x_n - \widehat{u}\|^2 - 2\gamma_n\langle x_n - \widehat{u}, x_{n-1} - x_n\rangle + \gamma_n^2\|x_{n-1} - x_n\|^2\}; \\ x_{n+1} = P_{C_n}(x_0). \end{cases} \tag{4.1}$$

Let a sequence $\{x_n\}_{n=0}^{\infty}$ in $\mathcal{H}$ be generated by (4.1), for each $n \geq 1$, where$\{\mu_n\} \subset (0, \frac{2}{\rho})$, $\{\gamma_n\} \subset [0, \gamma]$, $\gamma \in [0, 1)$, $\{r_n\} \subset (0, \infty)$, with $\alpha \in (0, \frac{1}{L})$ such that L is the spectral radius of h^*h and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$. Assume that the following conditions hold:

C1 $\sum_{n=1}^{\infty} \gamma_n\|x_n - x_{n-1}\| < \infty$;

C2 $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

C3 $\liminf_{n\to\infty} r_n > 0$;

C4 $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < \frac{2}{\rho} < 2\rho$.

Then the sequence $\{x_n\}$ generated by Theorem (4.1) strongly convergence to a point $\widehat{u} = P_\Gamma x_1$.

Split Feasibility Problem: Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and $h : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Let C and Q be closed, convex and nonempty subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The split feasibility problem (SFP) is the problem to find $\widehat{x} \in C$ such that $S\widehat{x} \in Q$. We represent the solution sets by $\omega := C \cap h^{-1}(Q) = \{\widehat{y} \in C : h\widehat{y} \in Q\}$. Censor and Elfving [15] introduced first time it, to solve inverse problems and their application to medical image reconstruction and radiation therapy and modeling and simulation in a finite dimensional Hilbert space. Recall C is the function

$$b_C(\widehat{x}) := \begin{cases} 0, & \widehat{x} \in C; \\ \infty, & \textit{otherwise}. \end{cases}$$

The proximal mapping of b_C is the metric projection on C,

$$\begin{aligned} prox_{b_C} &= \arg\min_{\widehat{p}\in C} \|\widehat{p} - \widehat{x}\| \\ &= P_C(\widehat{x}). \end{aligned}$$

Let $h : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator and h^* the adjoint of h. Let P_Q be the projection of $\mathcal{H}_2$ onto a nonempty, convex and closed subset Q. Take: $f(\widehat{x}) = \frac{1}{2}\|h\widehat{x} - P_Q h\widehat{x}\|^2$ and $g(\widehat{x}) = b_C(\widehat{x})$. Then, we compute the split feasibility problem from following theorem in strong convergence with inertial and split equilibrium problem form.

Corollary 4.2. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Let $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ be two bifunctions satisfying (A1)-(A4) of Assumption 2.3 such that G is upper semicontinuous. Let $h : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Let $A : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ be a maximally monotone operator and let $B : \mathcal{H}_1 \to \mathcal{H}_1$ be a monotone and ρ-Lipschitz operator for some $\rho > 0$. Assume that ω is the set of solutions of problem (2.1) and $\omega \neq 0$. Let γ_n be a bounded real sequence and $\mu \in (0, \frac{2}{\|h\|^2})$. Assume that $\Gamma = \omega \cap \Omega \neq \emptyset$, where $\Omega := \{x^* \in C : x^* \in EP(F) \quad \text{and} \quad hx^* \in EP(G)\}$. For given $x_0, x_1 \in \mathcal{H}_1$ let the iterative sequence $\{x_n\}$,$\{m_n\}$,$\{u_n\}$,$\{v_n\}$ and $\{w_n\}$ are generated by*

$$\begin{cases} m_n = x_n + \gamma_n(x_n - x_{n-1}); \\ u_n = \beta_n m_n + (1-\beta_n) T^F_{r_n}(I - \alpha h^*(I - T^G_{r_n})h)m_n; \\ v_n = P_C(I - \mu_n h^*(I - P_Q)h)u_n; \\ w_n = v_n - \mu_n(h^*(I - P_Q)hv_n - h^*(I - P_Q)hu_n); \\ C_{n+1} = \{\widehat{u} \in C_n : \|w_n - \widehat{u}\|^2 \leq \|x_n - \widehat{u}\|^2 - 2\gamma_n\langle x_n - \widehat{u}, x_{n-1} - x_n\rangle + \gamma_n^2\|x_{n-1} - x_n\|^2\}; \\ x_{n+1} = P_{C_n}(x_0). \end{cases} \tag{4.2}$$

Let a sequence $\{x_n\}_{n=0}^{\infty}$ in $\mathcal{H}$ be generated by (4.2), for each $n \geq 1$, where$\{\mu_n\} \subset (0, \frac{2}{\|h\|^2})$, $\{\gamma_n\} \subset [0, \gamma]$, $\gamma \in [0, 1)$, $\{r_n\} \subset (0, \infty)$, with $\alpha \in (0, \frac{1}{L})$ such that L is the spectral radius of h^*h and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$. Assume that the following conditions hold:

C1 $\sum_{n=1}^{\infty} \gamma_n \|x_n - x_{n-1}\| < \infty$;

C2 $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

C3 $\liminf_{n\to\infty} r_n > 0$;

C4 $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < \frac{2}{\|h\|}$.

Then the sequence $\{x_n\}$ generated by Theorem (4.2) strongly convergence to a point $\widehat{u} = P_\Gamma x_1$.

5. Example and Numerical Results

This section shows effectiveness to our algorithm by following given examples and numerical results.

Example 5.1. Let $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y\rangle = xy$, for all $x, y \in \mathbb{R}$ and induced usual norm $|\cdot|$. Let $F : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is defined as $F(x, y) = 2x(y - x)$ where $x, y \in F$ and let $G : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is defined as $G(u, v) = u(v - u)$ where $u, v \in G$. For $r > 0$, we define three mappings $h, A, B : \mathbb{R} \to \mathbb{R}$ are defined as $h(x) = 3x$, $Ax = 4x$ and $Bx = 3x$, respectively. For all $x = x_0, x_1 \in \mathbb{R}$ and B be a monotone and ρ-Lipschitz operator for some $\rho > 0$ and A is maximal monotone. Then there exist unique sequences $\{x_n\}$,$\{m_n\}$,$\{u_n\}$,$\{v_n\}$ and $\{w_n\}$ are generated by iterative method in theorem (3.1). Choose $\alpha = 0.5$,$\beta = \frac{n}{100n-1}$, $r = \frac{n}{100n-1}$, $L = 3$ and μ=0.004.

Since $\gamma_n = \begin{cases} \min\{\frac{1}{n^2\|x_n - x_{n-1}\|}, 0.5\} & \textit{if} \quad x_n \neq x_{n-1}; \\ 0.5 & \text{otherwise,} \end{cases}$

then $\{x_n\}$ conveges strongly.

It is easy to prove that the bifunction F and G satisfy the A_1-A_2 and G is upper semicontinuous. h is bounded linear operator on $\mathbb{R}$ with adjoint opertator h^* and $\|h\| = \|h^*\| = 3$. Moreover, $Sol(EP(F) = \{0\}$, $Sol(EP(G)) = \{0\}$. Hence $\Gamma = (A + B)^{-1}(0) \cap \Omega = 0$. Now, we solved this numerical example in six step,

Step 1. Find $z \in Q$ such that $G(z, y) + \frac{1}{r}\langle y - z, z - hx\rangle \geq 0$ for all $y \in Q$.

$$\begin{aligned} G(z, y) + \frac{1}{r}\langle y - z, z - hx\rangle \geq 0 \quad &\Leftrightarrow \quad z(y - z) + \frac{1}{r}\langle y - z, z - hx\rangle \geq 0, \\ &\Leftrightarrow \quad rz(y - z) + (y - z)(z - hx) \geq 0, \\ &\Leftrightarrow \quad (y - z)((1 + r)z - hx) \geq 0 \end{aligned}$$

for all $y \in Q$. Thus, By Lemma 2.4(2), we know that $T_r^G hx$ is single-valued for each $x \in C$. Hence $z = \frac{hx}{1+r}$.

Step 2. Find $m \in C$ such that $m = x - \alpha h^*(I - T_r^G)hx$. From Step 1, we get,

$$\begin{aligned} m = x - \alpha h^*(I - T_r^G)hx \quad &= \quad x - \alpha h^*(I - T_r^G)hx, \\ &= \quad x - \alpha(3x - \frac{3(hx)}{1 + r}), \\ &= \quad (1 - 3\alpha)x + \frac{3\alpha}{1 + r}(hx). \end{aligned}$$

Step 3.
Find $u \in C$ such that $F(u,v) + \frac{1}{r}\langle v-u, u-m\rangle \geq 0$ for all $v \in C$. From Step Two, we have

$$\begin{aligned} F(u,v) + \frac{1}{r}\langle v-u, u-m\rangle \geq 0 \quad &\leftrightarrow \quad (2u)(v-u) + \frac{1}{r}\langle v-u, u-m\rangle \geq 0,\\ &\leftrightarrow \quad r(2u)(v-u) + (v-u)(u-m) \geq 0,\\ &\leftrightarrow \quad (v-u)((1+2r)u-m) \geq 0 \end{aligned}$$

for all $v \in C$. Similarly, by Lemma 2.4(2), we obtain $u = \frac{m}{1+2r} = \frac{(1-3\alpha)x}{1+2r} + \frac{3\alpha hx}{(1+r)(1+2r)}$.

Step 4.
Formulations for the sequences.

$$\begin{cases} x_0 = x \in \mathbb{R};\\ m_n = x_n + \gamma_n(x_n - x_{n-1});\\ u_n = \frac{n}{100n+1}m_n + (1 - \frac{n}{100n+1})(\frac{(1-3\alpha)x_n}{1+2r} + \frac{3\alpha hx_n}{(1+r)(1+2r)})m_n,\\ v_n = (\frac{1-3s}{1+4s}x_n - \frac{s}{1+4s}3x_n)u_n,\\ w_n = v_n - 0.004(3v_n - 3u_n); \end{cases}$$

Step 5.
Find $C_{n+1} = \{\widehat{u} \in C_n : \|w_n - \widehat{u}\|^2 \leq \|x_n - \widehat{u}\|^2 - 2\gamma_n\langle x_n - \widehat{u}, x_{n-1} - x_n\rangle + \gamma_n^2\|x_{n-1} - x_n\|^2\}$.
Since $\|w_n - \widehat{u}\|^2 \leq \|x_n - \widehat{u}\|^2 - 2\gamma_n\langle x_n - \widehat{u}, x_{n-1} - x_n\rangle + \gamma_n^2\|x_{n-1} - x_n\|^2\}$ we have

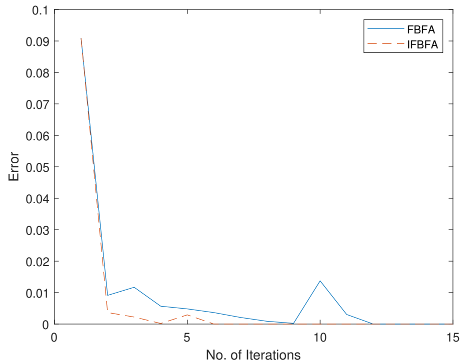
$$\frac{w_n + x_n}{2} \leq \widehat{u}.$$

Step 6.
Compute the numerical results of $x_{n+1} = P_{C_{n+1}}x_1$.
We provide a numerical test of a comparison between our inertial forward-backward-forward method defined in Theorem 3.2 (i.e $\gamma_n \neq 0$)and standard forward-backward-forward method (i.e $\gamma_n = 0$). The stoping criteria is defined as $E_n = \|x_{n+1} - x_n\| < 10^{-6}$. The different choices of x_0 and x_1 are giving as following:

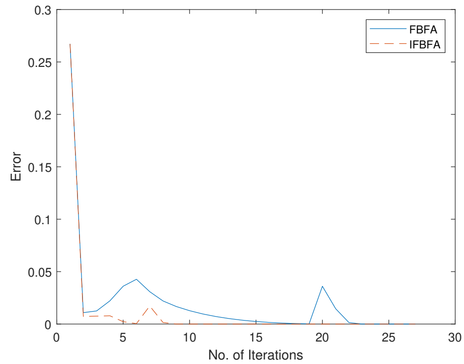
Table 1. Numerical results for Example 5.1

	No. of Iter.		**CPU(Sec)**	
	$\gamma_n = 0$	$\gamma_n \neq 0$	$\gamma_n = 0$	$\gamma_n \neq 0$
Choice 1. $x_0 = (2)$ and $x_1 = (-2)$	18	11	0.044675	0.041939
Choice 2. $x_0 = (10)$ and $x_1 = (2)$	29	13	0.051017	0.042856
Choice 3. $x_0 = (1.5)$ and $x_1 = (2.5)$	24	10	0.043589	0.037694

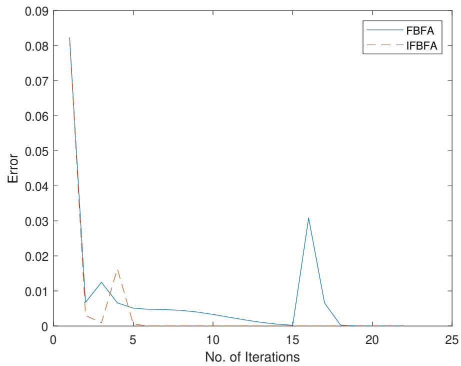
The error plotting E_n of $\gamma_n \neq 0$ and $\gamma_n = 0$ for each choices in Table 1. is shown in figure 1-3, respectively,



(a) Choice 1



(b) Choice 2



(c) Choice 3

Fig. 1. Evaluation of iterations for IFBFA and FBFA in Choice 3 of Example 5.1

Conclusion. The main aim of this paper is to propose an iterative algorithm to find an element for solving a class of split equilibrium problem and inclusion problem in Hilbert spaces. We introduce a modified inertial forward-backward-forward splitting algorithm and its convergence theorem for the split equilibrium problem and Inclusion problem in Hilbert spaces. We also proved there convergence and designed the algorithms by combining the forward-backward-forward splitting method and the shrinking projection method. Some applications and numerical example and computational results are implemented for bifunctions, which are generalized from the split equilibrium problem to illustrate the convergence which are presented in this paper.

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Accelerated Hybrid Mann-type Algorithm for Fixed Point and Variational Inequality Problems

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ABSTRACT

The purpose of this paper is to establish and study an accelerated hybrid Mann-type algorithm for the fixed point of nonexpansive mappings and variational inequality problems of monotone operators with the Lipschitz condition. Based on the Mann algorithm that generates a new iterative vector by a convex combination of the previous two iterative vectors, the advantageous behavior in the construction of a new iterative vector was observed due to the convex combination of three iterative vectors. Furthermore, by combining with the method known as the inertial Tseng's extragradient method, the accelerated hybrid Mann-type algorithm was established. To demonstrate the efficiency and advantages of this new algorithm, we have created some numerical results to compare the advantages of different areas with the previous existing results.

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1. Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$. Let C be a nonempty closed convex subset in H. Let $U : H \to H$ be a mapping. A point $x^* \in H$ is called a fixed point of U if $Ux^* = x^*$. The set of fixed points of U is denoted by $Fix(U)$.

A mapping $U : H \to H$ is said to be nonexpansive if $\|Ux - Uy\| \le \|x - y\|$ for all $x, y \in H$. A mapping $U : H \to H$ with $Fix(U) \neq \emptyset$ is said to be quasi-nonexpansive if $\|Ux - p\| \le \|x - p\|$ for all $x \in H$ and $p \in Fix(U)$.

Iterative method of fixed points of quasi-nonexpansive mappings has been studied and extended by many authors (see, for example, [9–12, 26, 33]). Notice that every nonexpansive mapping with a nonempty set of fixed points is a quasi-nonexpansive. It is well know that the fixed point problem for the mapping $U : H \to H$ is as follows:

$$\text{Find } x^* \in H \text{ such that } Ux^* = x^*.$$

Most of the problems in nonlinear analysis can be changed the forms to be the problems of finding a fixed point of a nonexpansive mapping and its generalizations. In 1953, Mann [22] created and introduced the explicit iteration procedure for a nonexpansive mapping as follows:

$$x_{n+1} = \mu_n x_n + (1 - \mu_n) U x_n, \quad n \ge 0, \tag{1.1}$$

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where $\{\mu_n\} \subseteq (0,1)$ satisfying $\sum_{n=1}^{\infty} \mu_n(1-\mu_n) = \infty$ if $Fix(U) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1.1) converges weakly to a fixed point of U.

Let $F : H \to H$ be an operator. The variational inequality problem (VIP) for F on C is to find a point $x^* \in C$ such that

$$\langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.2}$$

The solution set of VIP (1.2) is denoted by $VI(C,F)$. Variational inequality problems are fundamental in a broad range of mathematical and applied sciences; the theoretical and algorithmic foundations as well as the applications of variational inequality problems have been extensively studied in the literature and continue to attract intensive research, see for instance [2, 13, 18, 19, 23, 36, 37, 39] and the extensive list of references there in.

There are several methods for finding the a common solution of fixed point and variational inequality problem such as the projected gradient method, extragradient method, subgradient extragradient method. Many authors have discovered and introduced several iterative methods for solving VIP (1.2). One of the easiest methods is the following projection method, which can be seen as an extension of the projected gradient method for optimization problems:

$$x_{n+1} = P_C(x_n - \tau F x_n) \tag{1.3}$$

where P_C is denoted by the metric projection from H onto C. Convergence results for (1.3) need F to be Lipschitz continuous with Lipschitz constant L and $\alpha-$strongly monotone and $\tau \in (0, (2\alpha/L^2))$. In [16], He et al. showed that if the strong monotonicity assumption is relaxed to the monotonicity, then the projected gradient method may diverge. Note that, method (1.3) also works for strongly pseudo-monotone VIPs and co-coercive VIPs. To deal with the weakness of the method defined by (1.3). Korpelevich [20] proposed the extragradient method. The method is of the form:

$$x_0 \in C, \quad y_n = P_C(x_n - \tau F x_n), \quad x_{n+1} = P_C(x_n - \tau F y_n) \tag{1.4}$$

where $F : H \to H$ is $L-$Lipschitz continuous and monotone, $\tau \in (0, (1/L))$. Korpelevich showed that if $VI(C,F)$ is nonempty then the sequence $\{x_n\}$ generated by (1.4) converges weakly to an element of $VI(C,F)$. To see the variant forms of the method (1.4), the reader could refer to the recent papers of He et al. [17], Gárciga Otero and Iuzem [14], Solodov and Svaiter [28], Solodov [27]. Recently, Censor et al. [6–8] introduced the subgradient extragradient method as follows:

$$y_n = P_C(x_n - \tau F x_n), \quad x_{n+1} = P_{T_n}(x_n - \tau F y_n) \tag{1.5}$$

where $T_n = \{x \in H \mid x_n - \tau F x_n - y_n, x - y_n \leq 0\}$ and $\tau \in (0, (1/L))$. In method (1.5), they replaced two projections onto C by one projection onto C and one onto a half-space.

In [35], Tseng presented the extragradient method as follows:

$$y_n = P_C(x_n - \tau F x_n), \quad x_{n+1} = y_n - \tau(F y_n - F x_n). \tag{1.6}$$

The method (1.6) and subgradient extragradient method need only to compute one projection onto C in each update. Later, the method (1.6) has gained attention and popularity to solve VIP from many authors (see, e.g. [4, 30, 31, 34, 38] and the references therein).

In 2019, Thong and Hieu [32] introduced some Mann-type algorithms for variational inequality and fixed point problems. They obtained new theorems and good behavior of the numerical results. One of the interesting main theorems is stated as follows:

Theorem 3.1. Let $F : H \to H$ be a monotone and $L-$Lipchitz mapping on H. Assume that the sequence $\{\mu_n\} \subseteq [0, \mu]$, $\mu < \frac{1}{3}$ is non-decreasing and $\{\alpha_n\} \subseteq (\alpha, 0.5], \alpha > 0$ is a sequence of real numbers. Let $\lambda \in (0, (1/L))$ and $U : H \to H$ be a quasi-nonexpansive mapping such that $I - U$ is demiclosed at zero and $Fix(U) \cap VI(C, F) \neq \varnothing$. Let $x_0, x_1 \in H$, the sequence $\{x_n\}$ is defined by

$$\begin{cases} w_n = x_n + \mu_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda F w_n), \\ z_n = y_n - \lambda(F y_n - F w_n), \\ x_{n+1} = (1 - \alpha_n) w_n + \alpha_n U z_n. \end{cases}$$

Then the sequence $\{x_n\}$ converges weakly to an element of $Fix(U) \cap VI(C, F)$. Notice that the term $\mu_n(x_n - x_{n-1})$ is called an inertial extrapolation term by making use of the previous two iterates x_n and x_{n-1}. The inertial extrapolation term $\mu_n(x_n - x_{n-1})$ is employed in algorithm for the purpose of speeding up the rate of convergence of the algorithm. The vector $(x_n - x_{n-1})$ is acting as an impulsion term and μ_n is acting as a speed regulator (see, e.g. [21, 25]).

On the other hand, for observing the above method especially for the last line updating, we found an anonymous example in the Euclidean space $\mathbb{R}^2$ that provides some advantage geometrical structures of the convex combination of the previous three iterative vectors; w_n, z_n, Uz_n, for updating the new iterative vector x_{n+1}. It can be illustrated via the figures as below:

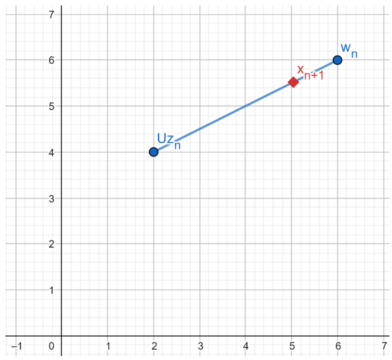

Fig. 1. x_{n+1} lies on a straight line formed by a convex combination of two vectors w_n and Uz_n.

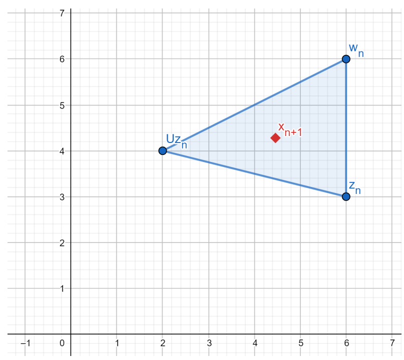

Fig. 2. x_{n+1} lies on a triangle formed by a convex combination of three vectors w_n, z_n and Uz_n.

It is explained by the visual indication of the geometric structure from Figure 1 and Figure 2 that the new vector x_{n+1} that obtained form the convex combination of three iterative vectors is likely to provide better performance than the convex combination of two iterative vectors.

Motivated by the directions mentioned above, in this paper, we aim to introduce and study a new accelerated hybrid Mann-type algorithm by using the convex combination of three iterative vectors for finding a solution of fixed point and variational inequality problems in the framework of Hilbert spaces. Further, we intend to establish some numerical experiments to illustrate the behavior of the new obtained algorithm. For representing the advantage of the main results, we have created some numerical results to compare advantages of different areas with the previous existing results.

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. The weak convergence of $\{x_n\}_{n=1}^{\infty}$ to x is denoted by $x_n \rightharpoonup x$ as $n \to \infty$ while the strong convergence of $\{x_n\}_{n=1}^{\infty}$ to x is written as $x_n \to x$ as $n \to \infty$. For each $x, y, z \in H$ and $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$\alpha + \beta + \gamma = 1$, we have

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2. \tag{2.1}$$

For each point $x \in H$, there exists the unique nearest point in C, denoted by $P_C x$ such that $\|x - P_C x\| = \inf_{y \in C} \|x - y\| \leq \|x - y\|, \quad \forall y \in C$. P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive.

Lemma 2.1. [3, 5, 15] *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$. Then $z = P_C x \Leftrightarrow \langle x - z, z - y\rangle \geq 0, \quad \forall y \in C$.*

Lemma 2.2. T31,T32,T33 *Let C be a closed and convex subset in a real Hilbert space H, $x \in H$. Then*

(1) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \quad \forall y \in C$;

(2) $\|P_C x - y\|^2 \leq \|x - y\|^2 - \|x - P_C x\|^2, \quad \forall y \in C$.

Definition 2.3. [3, 5, 15] Assume that $T : H \to H$ is a nonlinear operator with $Fix(T) \neq \emptyset$. Then $I - T$ is said to be demiclosed at zero if for any $\{x_n\}$ in H, the following implication holds:

$$x_n \rightharpoonup x \text{ and } (I - T)x_n \to 0 \quad \Rightarrow \quad x \in Fix(T).$$

Definition 2.4. [3, 5, 15] Let $T : H \to H$ be an operator. Then

- T is called $L-$Lipschitz continuous with $L > 0$ if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H.$$

- T is called monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0, \quad \forall x, y \in H.$$

Lemma 2.5. [1] *Let $\{\phi_n\}, \{\delta_n\}$ and $\{\alpha_n\}$ be sequences in $[0, +\infty)$ such that*

$$\phi_{n+1} \leq \phi_n + \alpha_n(\phi_n - \phi_{n-1}) + \delta_n, \quad \forall n \geq 1, \quad \sum_{n=1}^{+\infty} \delta_n < +\infty,$$

and there exists a real number α with $0 \leq \alpha_n \leq \alpha < 1$ for all $n \in \mathbb{N}$. Then the following hold:

(1) $\sum_{n=1}^{+\infty} [\phi_n - \phi_{n-1}]_+ < +\infty$, where $[t]_+ := \max\{t, 0\}$;

(2) there exist $\phi^* \in [0, +\infty)$ such that $\lim_{n \to +\infty} \phi_n = \phi^*$.

Definition 2.6. Let H be a real Hilbert space. Then the set

$$\{z \in H \mid \exists \{x_{n_k}\} \subseteq \{x_n\} \text{ such that } x_{n_k} \rightharpoonup z\}$$

is called the set of all **sequential weak cluster point** of $\{x_n\}$.

Lemma 2.7. [24] *Let C be a nonempty set of H and $\{x_n\}$ be a sequence in H such that the following two conditions hold:*

(1) for every $x \in C$, $\lim_{n \to \infty} \|x_n - x\|$ exists;

(2) every sequential weak cluster point of $\{x_n\}$ is in C. Then $\{x_n\}$ converges weakly to a point in C.

Lemma 2.8. [29] *Assume that $F : C \to H$ is a continuous and monotone operator. Then x^* is a solution of (1.2) if and only if x^* is a solution of the following problem:*

$$\textit{find } x \in C \textit{ such that } \langle Fy, y - x\rangle \geq 0, \quad \forall y \in C.$$

3. Main Results

In this section, we introduce the new Mann-type algorithm called the accelerated hybrid Mann-type algorithm for solving some fixed point problems of a quasi-nonexpansive mapping and variational inequality problems of a monotone and $L-$Lipchitz mapping in the frame work of real Hilbert spaces.

Theorem 3.1. *Let $F : H \to H$ be a monotone and $L-$Lipchitz mapping on H. Assume that the sequence $\{\mu_n\} \subseteq [0, \mu]$, $\mu < \frac{1}{5}$ is non-decreasing, $\{\alpha_n\} \subseteq (\alpha, 0.5], \alpha > 0$, $\{\beta_n\} \subseteq [0, 0.5]$ and $\{\gamma_n\} \subseteq [0.5, 1)$ is a sequence of real numbers. Let $\lambda \in (0, (1/L))$ and $U : H \to H$ be a quasi-nonexpansive mapping such that $I - U$ is demiclosed at zero and $Fix(U) \cap VI(C, F) \neq \varnothing$. Let $x_0, x_1 \in H$, the sequence $\{x_n\}$ is defined by*

$$\begin{cases} w_n = x_n + \mu_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda F w_n), \\ z_n = y_n - \lambda(F y_n - F w_n), \\ x_{n+1} = \gamma_n w_n + \beta_n z_n + \alpha_n U z_n, \end{cases} \tag{3.1}$$

where $\alpha_n + \beta_n + \gamma_n = 1$. Then the sequence $\{x_n\}$ converges weakly to an element of $Fix(U) \cap VI(C, F)$.

Proof. We split the proof into three claims. Let $x^* \in Fix(U) \cap VI(C, F)$.
Claim 1.

$$\|z_n - x^*\|^2 \leq \|w_n - x^*\|^2 - (1 - \lambda^2 L^2)\|y_n - w_n\|^2, \quad \forall n \in \mathbb{N}. \tag{3.2}$$

We have

$$\begin{aligned} \|z_n - x^*\|^2 &= \|y_n - \lambda(F y_n - F w_n) - x^*\|^2 \\ &= \|y_n - x^*\|^2 + \lambda^2\|F y_n - F w_n\|^2 - 2\lambda\langle y_n - x^*, F y_n - F w_n\rangle \\ &= \|w_n - x^*\|^2 + \|w_n - y_n\|^2 + 2\langle y_n - w_n, w_n - x^*\rangle \\ &\quad + \lambda^2\|F y_n - F w_n\|^2 - 2\lambda\langle y_n - x^*, F y_n - F w_n\rangle \\ &= \|w_n - x^*\|^2 + \|w_n - y_n\|^2 - 2\langle y_n - w_n, y_n - w_n\rangle \\ &\quad + 2\langle y_n - w_n, y_n - x^*\rangle + \lambda^2\|F y_n - F w_n\|^2 \\ &\quad - 2\lambda\langle y_n - x^*, F y_n - F w_n\rangle \\ &= \|w_n - x^*\|^2 - \|w_n - y_n\|^2 + 2\langle y_n - w_n, y_n - x^*\rangle \\ &\quad + \lambda^2\|F y_n - F w_n\|^2 - 2\lambda\langle y_n - x^*, F y_n - F w_n\rangle. \end{aligned} \tag{3.3}$$

Since $y_n = P_C(w_n - \lambda F w_n)$, we get

$$\langle y_n - w_n + \lambda F w_n, y_n - x^*\rangle \leq 0,$$

equivalently

$$\langle y_n - w_n, y_n - x^*\rangle \leq -\lambda\langle F w_n, y_n - x^*\rangle. \tag{3.4}$$

Combining (3.3) and (3.4), we obtain

$$\begin{aligned} \|z_n - x^*\|^2 \leq\ & \|w_n - x^*\|^2 - \|w_n - y_n\|^2 - 2\lambda\langle F w_n, y_n - x^*\rangle \\ & + \lambda^2\|F y_n - F w_n\|^2 - 2\lambda\langle y_n - x^*, F y_n - F w_n\rangle \\ =\ & \|w_n - x^*\|^2 - \|w_n - y_n\|^2 + \lambda^2\|F y_n - F w_n\|^2 - 2\lambda\langle y_n - x^*, F y_n\rangle \end{aligned}$$

$$\begin{aligned}
&\leq \|w_n - x^*\|^2 - \|w_n - y_n\|^2 + \lambda^2 L^2 \|y_n - w_n\|^2 \\
&\quad - 2\lambda\langle y_n - x^*, Fy_n - Fx^*\rangle - 2\lambda\langle y_n - x^*, Fx^*\rangle \\
&\leq \|w_n - x^*\|^2 - (1 - \lambda^2 L^2)\|y_n - w_n\|^2. \qquad (3.5)
\end{aligned}$$

Claim 2.

$$\lim_{n\to\infty} \|Uz_n - z_n\| = 0. \qquad (3.6)$$

From (3.2), we have

$$\|z_n - x^*\| \leq \|w_n - x^*\|. \qquad (3.7)$$

Consider $\|x_{n+1} - w_n\|^2 = \|\gamma_n w_n + \beta_n z_n + \alpha_n U z_n - w_n\|^2$ and (2.1), we have

$$\begin{aligned}
\|x_{n+1} - w_n\|^2 &= \|\gamma_n w_n + \beta_n z_n + \alpha_n U z_n - w_n\|^2 \\
&= \|\gamma_n(w_n - w_n) + \beta_n(z_n - w_n) + \alpha_n(Uz_n - w_n)\|^2 \\
&\leq \beta_n\|z_n - w_n\|^2 + \alpha_n\|Uz_n - w_n\|^2.
\end{aligned}$$

Using (2.1) and (3.7) we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\gamma_n w_n + \beta_n z_n + \alpha_n U z_n - x^*\|^2 \\
&= \|\gamma_n(w_n - x^*) + \beta_n(z_n - x^*) + \alpha_n(Uz_n - x^*)\|^2 \\
&= \gamma_n\|w_n - x^*\|^2 + \beta_n\|z_n - x^*\|^2 + \alpha_n\|Uz_n - x^*\|^2 \\
&\quad - \gamma_n\beta_n\|z_n - w_n\|^2 - \gamma_n\alpha_n\|Uz_n - w_n\|^2 - \beta_n\alpha_n\|Uz_n - z_n\|^2 \\
&\leq \gamma_n\|w_n - x^*\|^2 + \beta_n\|z_n - x^*\|^2 + \alpha_n\|z_n - x^*\|^2 \\
&\quad - \gamma_n\beta_n\|z_n - w_n\|^2 - \gamma_n\alpha_n\|Uz_n - w_n\|^2 - \beta_n\alpha_n\|Uz_n - z_n\|^2 \\
&\leq \gamma_n\|w_n - x^*\|^2 + \beta_n\|w_n - x^*\|^2 + \alpha_n\|w_n - x^*\|^2 \\
&\quad - \gamma_n\beta_n\|z_n - w_n\|^2 - \gamma_n\alpha_n\|Uz_n - w_n\|^2 - \beta_n\alpha_n\|Uz_n - z_n\|^2 \\
&= \|w_n - x^*\|^2 - \gamma_n\beta_n\|z_n - w_n\|^2 - \gamma_n\alpha_n\|Uz_n - w_n\|^2 \\
&\quad - \beta_n\alpha_n\|Uz_n - z_n\|^2 \\
&\leq \|w_n - x^*\|^2 - \gamma_n\beta_n\|z_n - w_n\|^2 - \gamma_n\alpha_n\|Uz_n - w_n\|^2 \\
&= \|w_n - x^*\|^2 - \gamma_n\left(\beta_n\|z_n - w_n\|^2 + \alpha_n\|Uz_n - w_n\|^2\right) \\
&\leq \|w_n - x^*\|^2 - \gamma_n\|x_{n+1} - w_n\|^2. \qquad (3.8)
\end{aligned}$$

Moreover

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|(1 + \mu_n)(x_n - x^*) - \mu_n(x_{n-1} - x^*)\|^2 \\
&= (1 + \mu_n)\|x_n - x^*\|^2 - \mu_n\|x_{n-1} - x^*\|^2 + \mu_n(1 + \mu_n)\|x_n - x_{n-1}\|^2. \qquad (3.9)
\end{aligned}$$

We also have

$$\begin{aligned}
\|x_{n+1} - w_n\|^2 &= \|x_{n+1} - x_n - \mu_n(x_n - x_{n-1})\|^2 \\
&= \|x_{n+1} - x_n\|^2 + \mu_n^2\|x_n - x_{n-1}\|^2 - 2\mu_n\langle x_{n+1} - x_n, x_n - x_{n-1}\rangle \\
&\geq \|x_{n+1} - x_n\|^2 + \mu_n^2\|x_n - x_{n-1}\|^2 - 2\mu_n\|x_{n+1} - x_n\|\,\|x_n - x_{n-1}\| \\
&\geq (1 - \mu_n)\|x_{n+1} - x_n\|^2 + (\mu_n^2 - \mu_n)\|x_n - x_{n-1}\|^2. \qquad (3.10)
\end{aligned}$$

Combining (3.8), (3.9) and (3.10) we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq (1+\mu_n)\|x_n - x^*\|^2 - \mu_n\|x_{n-1} - x^*\|^2 + (1+\mu_n)\mu_n\|x_n - x_{n-1}\|^2 \\
&\quad - \gamma_n(1-\mu_n)\|x_{n+1} - x_n\|^2 - \gamma_n(\mu_n^2 - \mu_n)\|x_n - x_{n-1}\|^2 \\
&= (1+\mu_n)\|x_n - x^*\|^2 - \mu_n\|x_{n-1} - x^*\|^2 - \gamma_n(1-\mu_n)\|x_{n+1} - x_n\|^2 \\
&\quad (\mu_n + \mu_n^2 - \gamma_n\mu_n^2 + \gamma_n\mu_n)\|x_n - x_{n-1}\|^2 \\
&\leq (1+\mu_n)\|x_n - x^*\|^2 - \mu_n\|x_{n-1} - x^*\|^2 - \gamma_n(1-\mu_n)\|x_{n+1} - x_n\|^2 \\
&\quad ((1-\gamma_n)\mu_n^2 + (1+\gamma_n)\mu_n)\|x_n - x_{n-1}\|^2 \\
&\leq (1+\mu_n)\|x_n - x^*\|^2 - \mu_n\|x_{n-1} - x^*\|^2 \\
&\quad - \gamma_n(1-\mu_n)\|x_{n+1} - x_n\|^2 + 2\mu_n\|x_n - x_{n-1}\|^2 \\
&\leq (1+\mu_{n+1})\|x_n - x^*\|^2 - \mu_n\|x_{n-1} - x^*\|^2 \\
&\quad - \gamma_n(1-\mu_n)\|x_{n+1} - x_n\|^2 + 2\mu_n\|x_n - x_{n-1}\|^2. \qquad (3.11)
\end{aligned}$$

This follows that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 - \mu_{n+1}\|x_n - x^*\|^2 + 2\mu_{n+1}\|x_{n+1} - x_n\|^2 \\
&\quad \leq \|x_n - x^*\|^2 - \mu_n\|x_{n-1} - x^*\|^2 \\
&\qquad + 2\mu_n\|x_n - x_{n-1}\|^2 + 2\mu_{n+1}\|x_{n+1} - x_n\|^2 - \gamma_n(1-\mu_n)\|x_{n+1} - x_n\|^2.
\end{aligned}$$

Put $\Lambda_n := \|x_n - x^*\|^2 - \mu_n\|x_{n-1} - x^*\|^2 + 2\mu_n\|x_n - x_{n-1}\|^2$. We get

$$\Lambda_{n+1} - \Lambda_n \leq -(\gamma_n(1-\mu_n) - 2\mu_{n+1})\|x_{n+1} - x_n\|^2.$$

It follows from $\mu_n \leq \mu < \frac{1}{5}$ that $\gamma_n(1-\mu_n) - 2\mu_{n+1} \geq 0.5 - 2.5\mu > 0$. Therefore, we obtain

$$\Lambda_{n+1} - \Lambda_n \leq -\delta\|x_{n+1} - x_n\|^2 \leq 0 \qquad (3.12)$$

where $\delta = 0.5 - 2.5\mu$. This implies that the sequence $\{\Lambda_n\}$ is nonincreasing. And we have

$$\begin{aligned}
\Lambda_n &= \|x_n - x^*\|^2 - \mu_n\|x_{n-1} - x^*\|^2 + 2\mu_n\|x_n - x_{n-1}\|^2 \\
&\geq \|x_n - x^*\|^2 - \mu_n\|x_{n-1} - x^*\|^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|x_n - x^*\|^2 &\leq \mu_n\|x_{n-1} - x^*\|^2 + \Lambda_n \\
&\leq \mu\|x_{n-1} - x^*\|^2 + \Lambda_1 \\
&\leq \ldots \leq \mu^n\|x_0 - x^*\|^2 + \Lambda_1(\mu^{n-1} + \ldots + 1) \\
&\leq \mu^n\|x_0 - x^*\|^2 + \frac{\Lambda_1}{1-\mu}. \qquad (3.13)
\end{aligned}$$

We have

$$\begin{aligned}
\Lambda_{n+1} &= \|x_{n+1} - x^*\|^2 - \mu_{n+1}\|x_n - x^*\|^2 + 2\mu_{n+1}\|x_{n+1} - x^*\|^2 \\
&\geq -\mu_{n+1}\|x_n - x^*\|^2. \qquad (3.14)
\end{aligned}$$

From (3.13) and (3.14) we obtain

$$-\Lambda_{n+1} \leq \mu_{n+1}\|x_n - x^*\|^2 \leq \mu\|x_n - x^*\|^2 \leq \mu^{n+1}\|x_0 - x^*\|^2 + \frac{\mu\Lambda_1}{1-\mu}.$$

It follows from (3.12) that

$$\delta \sum_{n=1}^{k} \|x_{n+1} - x_n\|^2 \le \Lambda_1 - \Lambda_{k+1} \le \mu^{k+1}\|x_0 - x^*\|^2 + \frac{\Lambda_1}{1-\mu}$$
$$\le \|x_0 - x^*\|^2 + \frac{\Lambda_1}{1-\mu}.$$

This implies

$$\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|^2 < +\infty. \tag{3.15}$$

We obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.16}$$

We have

$$\begin{aligned} \|x_{n+1} - w_n\| &= \|x_{n+1} - x_n - \mu_n(x_n - x_{n-1})\| \le \|x_{n+1} - x_n\| + \mu_n \|x_n - x_{n-1}\| \\ &\le \|x_{n+1} - x_n\| + \mu \|x_n - x_{n-1}\|. \end{aligned} \tag{3.17}$$

From (3.16) and (3.17) we obtain

$$\lim_{n\to\infty} \|x_{n+1} - w_n\| = 0.$$

From (3.11) we get

$$\|x_{n+1} - x^*\|^2 \le (1+\mu_n)\|x_n - x^*\|^2 - \mu_n\|x_{n-1} - x^*\|^2 + 2\mu\|x_n - x_{n-1}\|^2. \tag{3.18}$$

By (3.15), (3.18) and Lemma 2.7 we have

$$\lim_{n\to\infty} \|x_n - x^*\| = l. \tag{3.19}$$

And by (3.9) we obtain

$$\lim_{n\to\infty} \|w_n - x^*\| = l. \tag{3.20}$$

We also have

$$0 \le \|x_n - w_n\| \le \mu \|x_n - x_{n-1}\| \to 0. \tag{3.21}$$

From

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\le \gamma_n\|w_n - x^*\|^2 + \beta_n\|z_n - x*\|^2 + \alpha_n\|Uz_n - x*\|^2 \\ &\le \gamma_n\|w_n - x^*\|^2 + \beta_n\|z_n - x^*\|^2 + \alpha_n\|z_n - x^*\|^2 \\ &= (1 - (\alpha_n + \beta_n))\|w_n - x^*\|^2 + (\alpha_n + \beta_n)\|z_n - x^*\|^2. \end{aligned}$$

This implies that

$$\begin{aligned} \|z_n - x^*\|^2 &\ge \frac{\|x_{n+1} - x^*\|^2 - \|w_n - x^*\|^2}{(\alpha_n + \beta_n)} + \|w_n - x^*\|^2 \\ &> \frac{\|x_{n+1} - x^*\|^2 - \|w_n - x^*\|^2}{\alpha} + \|w_n - x^*\|^2. \end{aligned} \tag{3.22}$$

It implies from (3.19), (3.20) and (3.22) that

$$\lim_{n\to\infty} \|z_n - x^*\|^2 \ge \lim_{n\to\infty} \|w_n - x^*\|^2 = l. \tag{3.23}$$

By (3.7) we get

$$\lim_{n\to\infty} \|z_n - x^*\|^2 \le \lim_{n\to\infty} \|w_n - x^*\|^2 = l. \tag{3.24}$$

Combining (3.23) and (3.24) we obtain

$$\lim_{n\to\infty} \|z_n - x^*\|^2 = l.$$

From (3.5) we have

$$(1-\lambda^2 L^2)\|y_n - w_n\|^2 \le \|w_n - x^*\|^2 - \|z_n - x^*\|^2.$$

This implies that

$$\lim_{n\to\infty} \|y_n - w_n\| = 0. \tag{3.25}$$

It also holds

$$\|z_n - y_n\| = \lambda \|Fy_n - Fw_n\| \le \lambda L \|y_n - w_n\| \to 0. \tag{3.26}$$

Combining (3.25) and (3.26) we obtain

$$\lim_{n\to\infty} \|z_n - w_n\| = 0. \tag{3.27}$$

From

$$Uz_n - w_n = \frac{1}{\alpha_n}(x_{n+1} - w_n - \beta_n(z_n - w_n))$$

we have

$$\|Uz_n - w_n\| = \left\|\frac{1}{\alpha_n}(x_{n+1} - w_n - \beta_n(z_n - w_n))\right\| \le \frac{1}{\alpha_n}\|x_{n+1} - w_n\| + \frac{\beta_n}{\alpha_n}\|z_n - w_n\|. \tag{3.28}$$

From $\alpha_n \ge \alpha$, it follows from (3.16), (3.27) and (3.28) that

$$\lim_{n\to\infty} \|Uz_n - w_n\| = 0. \tag{3.29}$$

Combining (3.27) and (3.29) we obtain

$$\|Uz_n - z_n\| \le \|Uz_n - w_n\| + \|z_n - w_n\| \to 0.$$

Claim 3. The sequence $\{x_n\}$ converges weakly to an element of $Fix(U) \cap VI(C,F)$. Indeed, since $\{x_n\}$ is a bounded sequence, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $z \in H$ such that $x_{n_k} \rightharpoonup z$. By (3.21) we get $w_{n_k} \rightharpoonup z$ and by (3.27) $z_{n_k} \rightharpoonup z$. It follows from (3.6) and demiclosedness of $I - U$ that $z \in Fix(U)$.

From $y_{n_k} = P_C(w_{n_k} - \lambda F w_{n_k})$ and F is monotone, we have for every $x \in C$ that

$$\begin{aligned}
0 &\le \langle y_{n_k} - w_{n_k} + \lambda F w_{n_k}, x - y_{n_k}\rangle \\
&= \langle y_{n_k} - w_{n_k}, x - y_{n_k}\rangle + \lambda \langle F w_{n_k}, x - y_{n_k}\rangle \\
&= \langle y_{n_k} - w_{n_k}, x - y_{n_k}\rangle + \lambda \langle F w_{n_k}, w_{n_k} - y_{n_k}\rangle + \lambda \langle F w_{n_k}, x - w_{n_k}\rangle \\
&\le \langle y_{n_k} - w_{n_k}, x - y_{n_k}\rangle + \lambda \langle F w_{n_k}, w_{n_k} - y_{n_k}\rangle + \lambda \langle Fx, x - w_{n_k}\rangle.
\end{aligned}$$

Passing to the limit, we get

$$\langle Fx, x - z\rangle \ge 0 \quad \forall x \in C.$$

By Lemma 2.8 we have $z \in VI(C,F)$. Therefore, we have shown that for every $x^* \in Fix(U) \cap VI(C,F)$, $\lim_{n\to\infty} \|x_n - x^*\|$ exists and each sequential weak cluster point of sequence $\{x_n\}$ is in $Fix(U) \cap VI(C,F)$. By Lemma 2.7 the sequence $\{x_n\}$ converges weakly to $z \in Fix(U) \cap VI(C,F)$. ∎

Corollary 3.2 (Thong and Hieu [32, Theorem 3.1]). *Let $F : H \to H$ be a monotone and $L-$Lipchitz mapping on H. Assume that the sequence $\{\mu_n\} \subseteq [0, \mu]$, $\mu < \frac{1}{5}$ is non-decreasing, $\{\alpha_n\} \subseteq (\alpha, 0.5]$, $\alpha > 0$ is a sequence of real numbers. Let $\lambda \in (0, (1/L))$ and $U : H \to H$ be a quasi-nonexpansive mapping such that $I - U$ is demiclosed at zero and $Fix(U) \cap VI(C, F) \neq \varnothing$. Let $x_0, x_1 \in H$ the sequence $\{x_n\}$ is defined by*

$$\begin{cases} w_n = x_n + \mu_n(x_n - x_{n-1}) \\ y_n = P_C(w_n - \lambda F w_n) \\ z_n = y_n - \lambda(F y_n - F w_n) \\ x_{n+1} = (1 - \alpha_n)w_n + \alpha_n U z_n. \end{cases} \tag{3.30}$$

Then the sequence $\{x_n\}$ converges weakly to an element of $Fix(U) \cap VI(C, F)$.

Proof. If we set $\beta_n = 0$ for all $n \in \mathbb{N}$, then $\gamma_n = 1 - \alpha_n$. Therefore, Theorem 3.1 can be reduced to Corollary 3.2 as required. ∎

4. Numerical Experiments

In this section, we compare the advantages of the new algorithm with the previous exiting algorithm introduced by Thong and Hieu [32, Theorem 3.1].

Example 4.1. [32] Let $H = \mathbb{R}$, $C = [-2, 5]$ and $F : \mathbb{R} \to \mathbb{R}$ be given by

$$Fx := x - 3 + \sin(x - 3)$$

and $U : \mathbb{R} \to \mathbb{R}$ be given by

$$Ux = \frac{x + 3}{\frac{x^2}{9} + 1} \quad \forall x \in \mathbb{R}.$$

The solution of the problem is $x^* = 3$. The stopping criterion is defined by Error $= \|x_{n+1} - x_n\| < 10^{-4}$. Choose $x_0 = 5$ and $x_1 = 4$. Figure 3 and figure 4 show a comparison of the numerical behavior of an accelerated hybrid Mann-type algorithm (3.1) with an advantage over Mann-type algorithm (3.30).

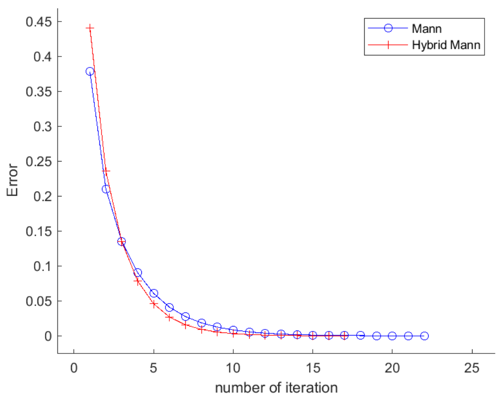

Fig. 3. Convergence behavior of $\{x_n\}$ of Example 4.1.

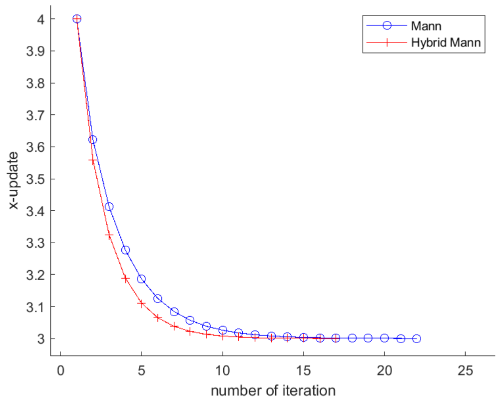

Fig. 4. $x-$update converges to solution x^* of Example 4.1.

Example 4.2. [32] Consider a nonlinear operator $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F(x, y) = (x + y + \sin x, -x + y + \sin y)$$

and the feasible set C is a box defined by $C = [-2, 5] \times [-2, 5]$. Let E be a 2×2 matrix defined by

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Mapping $U : \mathbb{R}^2 \to \mathbb{R}^2$ by $Uz = \|E\|^{-1}Ez$, where $z = (x, y)^T$. The solution of the problem is $x^* = (0, 0)^T$. The stopping criterion is defined by Error $= \|x_{n+1} - x_n\| < 10^{-4}$. Choose $x_0 = (7, 7)^T$ and $x_1 = (4, 3)^T$. By using this example, Figure 5 - Figure 9 show the advantage of (3.1) via numerical results.

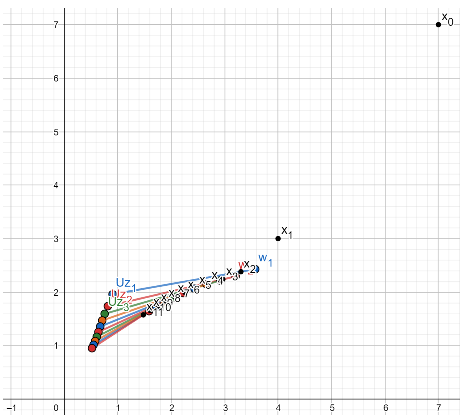

Fig. 5. The behavior of each x-update which lies on a straight line formed by a convex combination of two iterative vectors w_n and Uz_n.

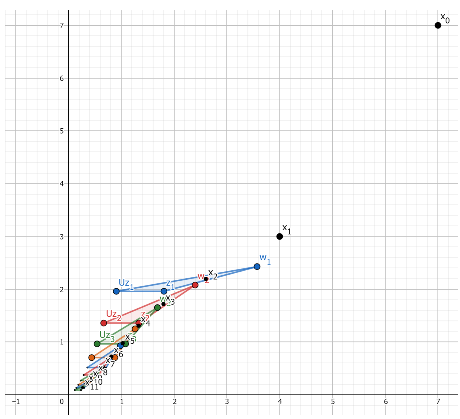

Fig. 6. The behavior of each x-update which lies on a triangle formed by a convex combination of three iterative vectors w_n, z_n and Uz_n.

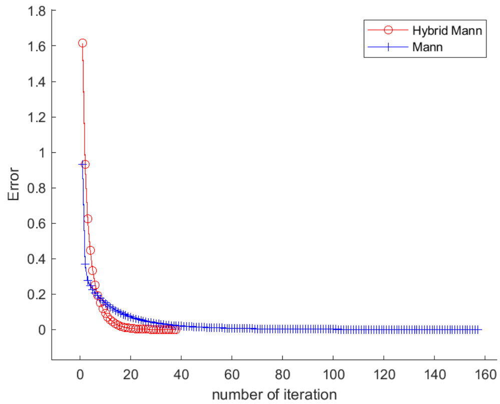

Fig. 7. Convergence behavior of $\{x_n\}$ of Example 4.2.

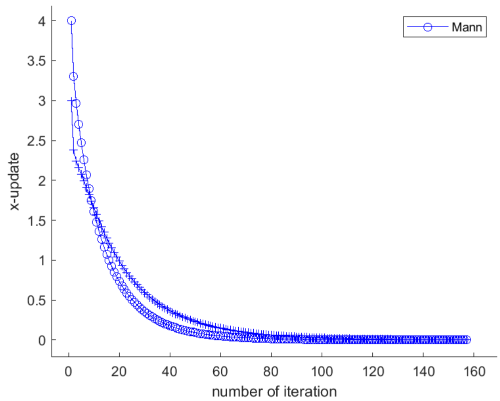

Fig. 8. Mann-type: x-update converges to solution $x^* = (0,0)^T$ of Example 4.2.

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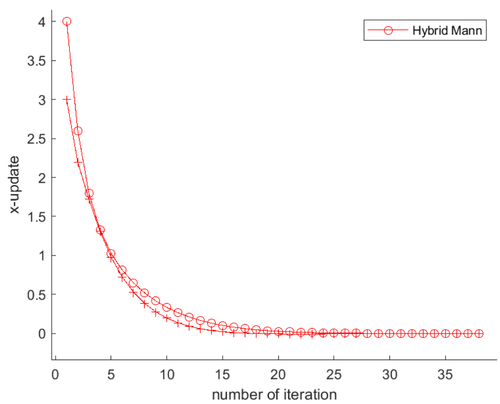

Fig. 9. Hybrid Mann-type: x-update converges to solution $x^* = (0,0)^T$ of Example 4.2.

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5. Conclusions

We introduced and studied the new Mann-type algorithm which is called the accelerated hybrid Mann-type algorithm and established the main theorem as follows:

Theorem 3.1. Let $F : H \to H$ be a monotone and $L-$Lipchitz mapping on H. Assume that the sequence $\{\mu_n\} \subseteq [0, \mu]$, $\mu < \frac{1}{5}$ is non-decreasing, $\{\alpha_n\} \subseteq (\alpha, 0.5], \alpha > 0$, $\{\beta_n\} \subseteq [0, 0.5]$ and $\{\gamma_n\} \subseteq [0.5, 1)$ is a sequence of real numbers. Let $\lambda \in (0, (1/L))$ and $U : H \to H$ be a quasi-nonexpansive mapping such that $I - U$ is demiclosed at zero and $Fix(U) \cap VI(C, F) \neq \varnothing$. Let $x_0, x_1 \in H$, the sequence $\{x_n\}$ is defined by

$$\begin{cases} w_n = x_n + \mu_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda F w_n), \\ z_n = y_n - \lambda(F y_n - F w_n), \\ x_{n+1} = \gamma_n w_n + \beta_n z_n + \alpha_n U z_n, \end{cases}$$

where $\alpha_n + \beta_n + \gamma_n = 1$. Then the sequence $\{x_n\}$ converges weakly to an element of $Fix(U) \cap VI(C, F)$.

The above theorem not only extends the theoretical concepts of the previous research work, but also provides numerical results that have an advantage over the previous work proposed by Thong and Hieu [32, Theorem 3.1]. It can be clearly seen in section 4 of this paper.

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